Nets of higher-dimensional cubes

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Abstract

In this extended abstract, we show that every ridge unfolding of an n-cube is without self-overlap, yielding a valid net. The results are obtained by developing machinery that translates cube unfolding into combinatorial frameworks. The bounding boxes of these cube nets are also explored using integer partitions.

1   Introduction

The study of unfolding polyhedra was popularized by Albrecht Dürer in the early 16th century in his influential book *The Painter's Manual*. It contains the first recorded examples of polyhedral *nets*, connected edge unfoldings of polyhedra that lay flat on the plane without overlap. Motivated by this, Shephard [6] conjectures that every convex polyhedron can be cut along certain edges and admits a net. This claim remains tantalizingly open.

We consider this question for higher-dimensional convex *polytopes*. The codimension-one faces of a polytope are *facets* and its codimension-two faces are *ridges*. The analog of an edge unfolding of polyhedron is the *ridge unfolding* of an n-dimensional polytope: the process of cutting the polytope along a collection of its ridges so that the resulting (connected) arrangement of its facets develops isometrically into an $\mathbb{R}^{n-1}$ hyperplane.

There is a rich history of higher-dimensional unfoldings of polytopes, with the collected works of Alexandrov [1] serving as seminal reading. In 1984, Turney [7] enumerates the 261 ridge unfoldings of the 4-cube, and in 1998, Buekenhout and Parker [2] extend this enumeration to the other five regular convex 4-polytopes. Both of these works focus on enumerative rather than geometric unfolding results. Miller and Pak [4] construct an algorithm which provides an unfolding of polytopes without overlap. However, their method allows cuts interior to facets, not just along ridges.

Our work targets ridge unfoldings of the n-cube. For the 3-cube, Figure 1 shows the 11 different unfoldings

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(up to symmetry), all of which yield nets. Section 2 generalizes this into our main result: *every* ridge unfolding of the n-cube results in a net. Section 3 considers packing these cube nets into boxes using integer partitions. Finally, we form a conjecture concerning regular convex polytopes in Section 4.
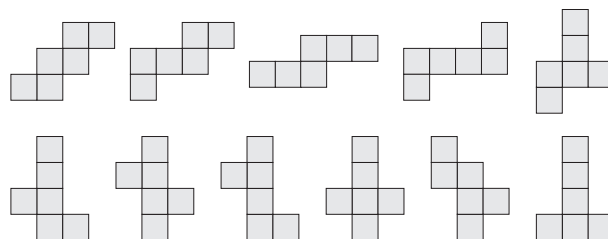

Figure 1: The 11 edge unfoldings of the 3-cube.

2   Rolling and Unfolding

We explore ridge unfoldings of a convex polytope P by focusing on the combinatorics of the arrangement of its facets in the unfolding. In particular, a ridge unfolding induces a tree whose nodes are the facets of the polytope and whose edges are the uncut ridges between the facets [5]. Indeed, this is a spanning tree in the 1-skeleton of the dual of P.

The dual of the n-cube is the n-orthoplex, whose 1-skeleton forms the n-*Roberts graph*. The $2n$ nodes of this graph (corresponding to the $2n$ facets of the n-cube) can arranged on a circle so that antipodal nodes represent opposite facets of the cube. Thus, unfoldings of an n-cube are in bijection with spanning trees of the n-Roberts graph.

Example 1 *Figure 2(a) considers an edge unfolding of the 3-cube with its underlying dual tree. This appears as a spanning tree on the 1-skeleton of the octahedral dual (b), redrawn on the 3-Roberts graph (c).*
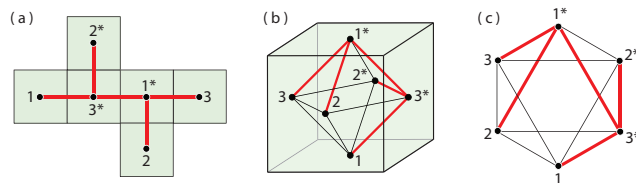

Figure 2: An unfolding of a 3-cube with its corresponding spanning tree on the 3-Roberts graph.

Recall that a ridge unfolding of an n-cube is a connected arrangement of its $2n$ facets, developed isometrically into hyperplane $\mathbb{R}^{n-1}$. Begin the unfolding by choosing a (base) facet b of the n-cube, placing it on the hyperplane. Then the normal vector n_b to b becomes normal to the hyperplane. Consider an adjacent facet c to b, and roll the cube along the ridge between these facets, with facet c now landing on the hyperplane. Figure 3 shows a rendering of the orthogonal projection of such a roll, with $c*$ and $b*$ corresponding to the antipodal facets of c and b, and the marked red edge representing the ridge between c and b.
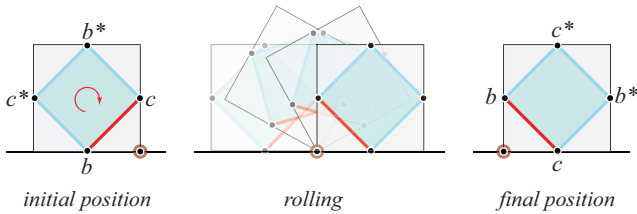

Figure 3: Rolling a cube on a hyperplane.

Since we rotate only along the plane spanned by the normal vectors n_b and n_c, the remaining directions stay fixed in the development. This is captured combinatorially as a rotation of a subgraph of the Roberts graph:

Definition 1 *A* roll *from base facet b towards an adjacent facet c rotates the four nodes $\{b, c, b*, c*\}$ of the Roberts graph along the quadrilateral (keeping the remaining nodes fixed), making c the new base facet.*

Figure 4 shows an example for the 5-cube, where the highlighted quadrilateral (depicting the roll) is invoking the colored square of Figure 3.
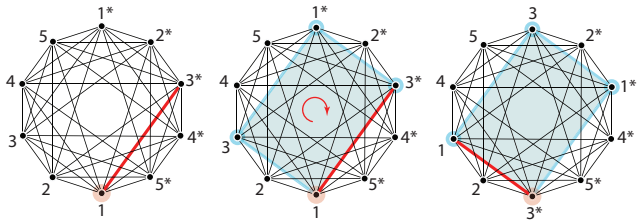

Figure 4: Rotating facet 1 towards 3^* on a 5-cube.

The advantage of unfolding a cube (compared to an arbitrary convex polytope) into hyperplane $\mathbb{R}^{n-1}$ is that its $(n-1)$-cube facets naturally tile this hyperplane. We exploit this by recasting the unfolding in the language of lattices. Without loss of generality, we can situate a ridge unfolding of the n-cube so that the centroid of each facet occupies a point of the integer lattice $\mathbb{Z}^{n-1}$ of $\mathbb{R}^{n-1}$. To see the lattice structure manifest in the n-Roberts graph, we imbue the latter with a *coordinate system*: arbitrarily label the $2n-2$ edges of the

n-Roberts graph incident to the base node with the directions

$$\{x_1, -x_1, x_2, -x_2, \ldots, x_{n-1}, -x_{n-1}\},$$

where edges incident to antipodal nodes get opposite directions.[1] Figure 5 shows examples of coordinate systems for the 3D, 4D, and 5D cases.
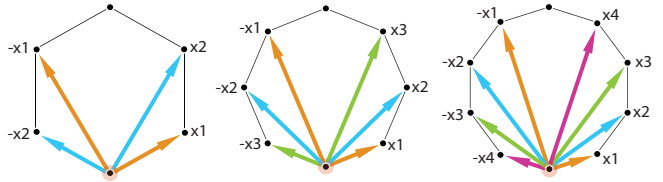

Figure 5: Coordinate systems for 3D, 4D, and 5D cubes.

These $n-1$ directions are mapped to the axes of the $\mathbb{R}^{n-1}$ hyperplane into which the n-cube unfolds. In particular, the $2n-2$ ridges of the n-cube incident to the base facet are in bijection with these coordinate directions, with opposite directions corresponding to parallel ridges of the facet. The roll keeps track of the combinatorics, whereas the coordinate system shows the direction of unfolding in the lattice. This is made precise:

Lemma 1 *Let T be a spanning tree of the n-Roberts graph with a coordinate system. The unfolding of the n-cube along T into $\mathbb{R}^{n-1}$ can be obtained by mapping T to the lattice $\mathbb{Z}^{n-1}$ through a sequence of rolls.*

Proof. Choose some base facet b of T and map it to some point $p_b \in \mathbb{Z}^{n-1}$. Let node c be adjacent to b along T with associated direction x from the coordinate system. The roll from b towards c maps node c to the point in $\mathbb{Z}^{n-1}$ that is adjacent to p_b in direction x. The four facet labels $\{b, c, b*, c*\}$ permute with the roll of the cube whereas the coordinate system directions are always anchored to the base facet. In particular, after the roll, facet $b*$ lies in the x direction with respect to the new base facet c, since the plane spanned by normal vectors n_b and n_c was rotated.

Given any node t of T, traverse the path between b and t through a series of rolls as described above; this maps all the nodes of T into $\mathbb{Z}^{n-1}$. To obtain the unfolding of the n-cube, simply replace each mapped point of the lattice with an $(n-1)$-cube. $\qquad\square$

Example 2 *Figure 6 shows an unfolding of the 3-cube along a spanning path using Lemma 1. At each iteration, there is a roll of the Roberts graph and a direction of unfolding based on the given coordinate system. The unfolded facets are colored white, and the unfolded ridges*

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[1] The isometry group of the cube acts simply transitively on these labelings. Thus, without loss of generality, we can choose a counterclockwise labeling of the edges in cyclic order.

become dashed-lines. Figure 7 showcases a 3-cube unfolding along a spanning tree. Given any two nodes of this tree, we unfold along the path between these nodes by rolls using Lemma 1. Figure 8 provides an example of an unfolding of the 4-cube along a spanning path.
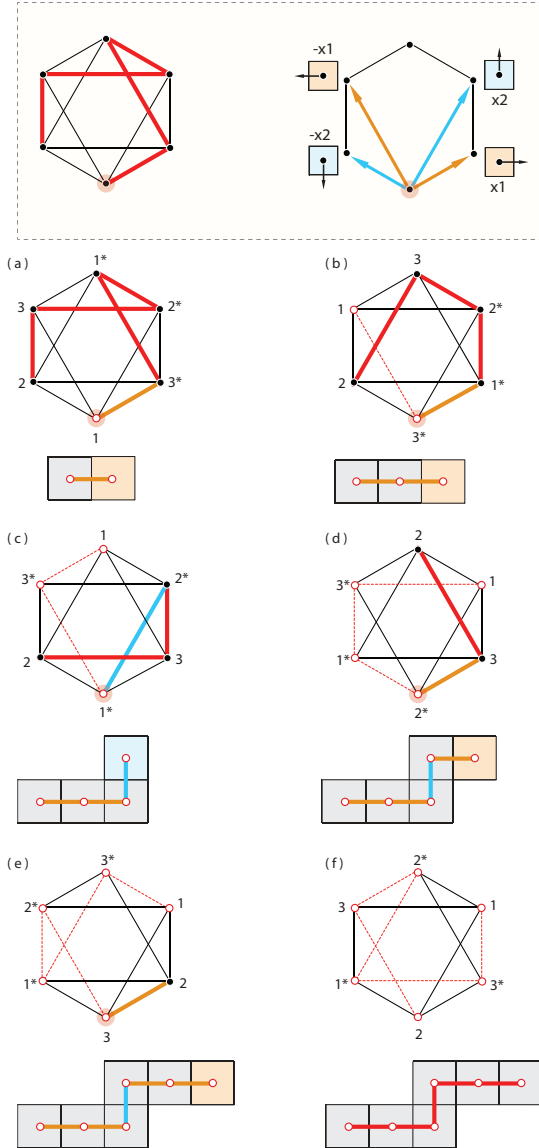

Figure 6: Unfolding a 3-cube along a spanning path.

Lemma 2 *Let T be a spanning tree of the n-Roberts graph with a coordinate system. If direction x is used in the unfolding along some path of T, direction $-x$ will not be used in the unfolding along this path.*

Proof. Assume we roll along a path in the x direction, moving the current base facet b into the $-x$ direction. Since b has now been visited, it cannot be used again in the unfolding. Thus, the only way a roll along direction
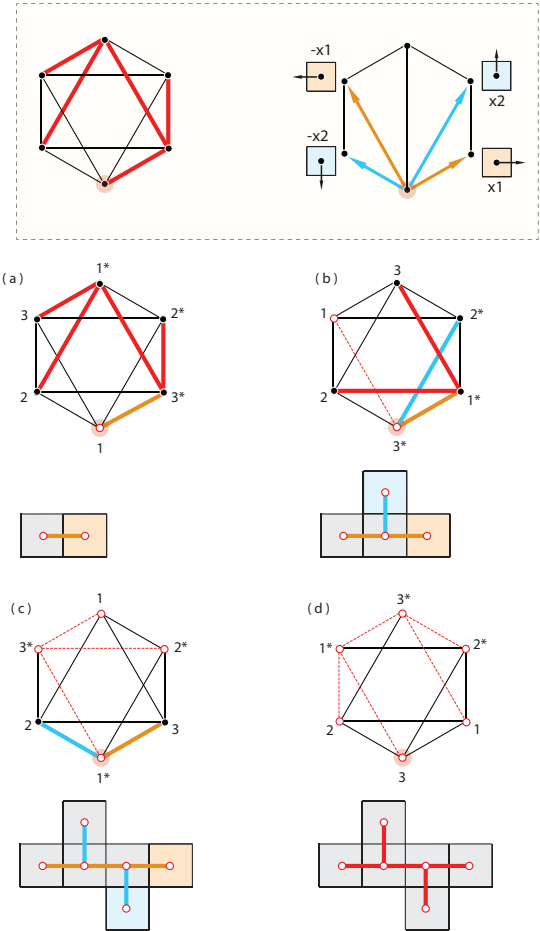

Figure 7: Unfolding a 3-cube along a spanning tree.

$-x$ can occur is if b is rotated out of that direction. However, the only moves that can displace b are rolls along the x and $-x$ directions. The latter is not possible and the former simply replaces b with another visited facet, continuing to obstruct motion in the $-x$ direction. $\square$

Example 3 *Figure 6(ab) shows an example where the first roll is in direction x_1, moving facet 1 into the $-x_1$ position, and facet 3^* into the base position. Since facet 1 has been visited, rolling in direction $-x_1$ is restricted. Another roll in Figure 6(bc) displaces 1 but simply replaces it with facet 3^*, which has now been visited.*

Theorem 3 *Every ridge unfolding of an n-cube yields a net.*

Proof. Consider an unfolding of the n-cube, given by a spanning tree T on the n-Roberts graph. By Lemma 2, antipodal directions will never appear in unfolding of paths. Thus, as the combinatorial distance between any two nodes of a path along the spanning tree T increases, the Euclidean distance of their respective facets in the hyperplane $\mathbb{R}^{n-1}$ (under the mapping to the integer lattice from Lemma 1) strictly increases. Since the facets
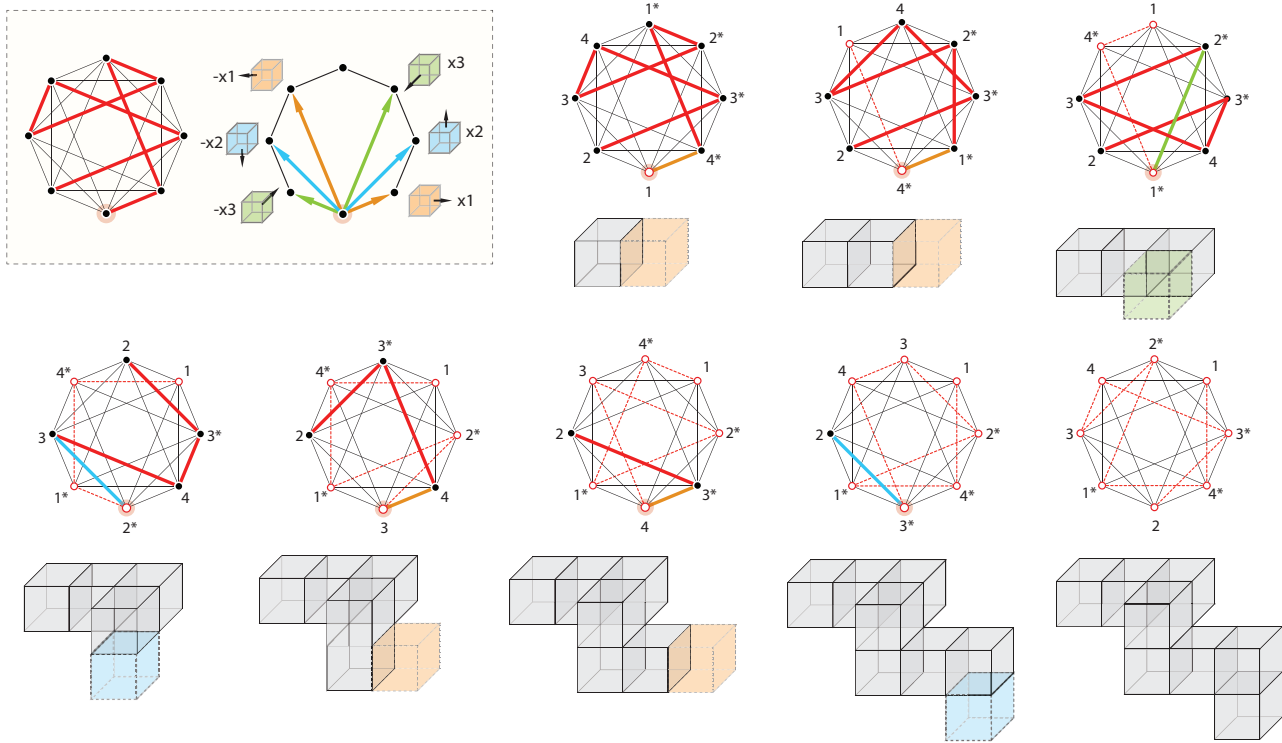
Figure 8: Unfolding a 4-cube along a spanning path.

in the unfolding along any path of T do not overlap, the unfolding of the entire tree T results in a net. □

3 Packings and Partitions

Having unfolded cubes into their nets, we now turn to packing these nets into boxes. A *box* (or *orthotope*) is the Cartesian product of intervals, and the *bounding box* of a net is the smallest box containing the net, with box sides parallel to the ridges of the net.

Definition 2 *An n-cube partition is an integer partition of $3n - 2$ into $n - 1$ parts, where each part is at least two.*
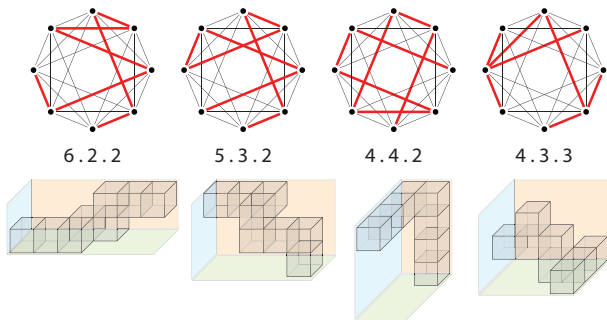

Figure 9: Spanning trees and bounding boxes.

Example 4 *Figure 9 displays four spanning trees of the 4-cube and their corresponding nets in bounding boxes. Notice that the dimensions of each bounding box form a 4-cube partition. In particular, these are all the possible 4-cube partitions. Theorem 4 below claims that all 261 nets of the 4-cube must fit into one of these four boxes.*

Theorem 4 *For every net of an n-cube, the dimensions of its bounding box is an n-cube partition.*

Proof. Each net of the n-cube has $2n$ facets that need to be unfolded in $\mathbb{R}^{n-1}$. Since each facet is an $(n-1)$-cube, the placement of the first facet in the unfolding contributes $n-1$ to the bounding box number of the net, one for each of its $n-1$ dimensions. We show that each of the remaining $2n-1$ facets of the unfolding increases the bounding box number by exactly 1, resulting in a total box number of $1 \cdot (n-1) + (2n-1) \cdot 1 = 3n-2$.

Suppose (by contradiction) that in the unfolding, the roll from facet b to adjacent facet c in direction x does not increase the bounding box number of the current net. Assume the ridge between b and c is supported by some hyperplane $\mathcal{H}$ of $\mathbb{R}^{n-1}$. Since the box number did not increase, there must be another facet (call it d) in the current unfolding that lies on the same side of hyperplane $\mathcal{H}$ as c. Thus, the unfolding of the path between facets c and d must have crossed $\mathcal{H}$ at least twice, moving along x in both the positive and negative directions, contradicting Lemma 2.
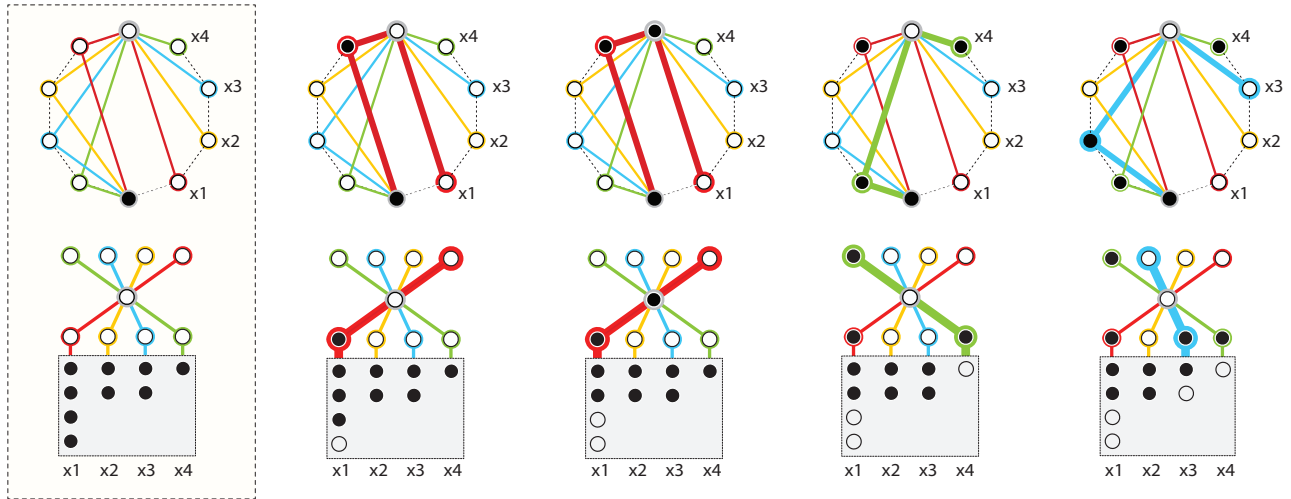
Figure 10: Rolls on the Roberts graphs reinterpreted as a token sliding game.

Finally, it needs to be shown that our cube will roll in all $n-1$ unfolding dimensions (satisfying the requirement that each part of a cube partition is a least two). But the cube net is a spanning tree of the Roberts graph, with the unfolding forced to visit all the nodes. And such visits can only be accomplished by rolling along each of the $n-1$ distinct directions. $\square$

The converse of Theorem 4 also holds: given an integer partition of $3n-2$ into $n-1$ parts, there exists an unfolding of an n-cube whose bounding box dimensions match the partition. The remainder of this section is devoted to proving this result. As discussed earlier, the placement of the first facet in the unfolding of the n-cube contributes $n-1$ to the bounding box number. Thus, the cube partition can be reinterpreted as an integer partition of $2n-1$ (the remaining facets) into $n-1$ parts (the possible directions), with each part at least one. For such a partition, our task is to find a sequence of rolls along the $n-1$ directions so that the $2n-1$ facets are unfolded into their respective partitioned directions. Without loss of generality, we consider rolls only in the positive directions.

In order to construct cube unfoldings for such partitions, we reinterpret the Roberts graph as a token sliding game, with Figure 10 serving as a Rosetta stone. Consider the first column of this figure, where the n-Roberts graph on top is unraveled below into a game board with $n-1$ slides (appropriately color-coded). Here, the base node of the Roberts graph is replaced by our given partition, one for each direction, with the $2n-1$ positions represented by black tokens. The goal of this game is to move these tokens into the $2n-1$ empty slots on the game board above by a sequence of slides, corresponding to rolls of the Roberts graph.

The top row of Figure 10 shows a 5-cube rolling twice in the x_1 direction, followed by a roll in the x_4 direction, and a roll in the x_3 direction. The bottom row displays the corresponding tokens moving along their appropriate slides, leaving the partition box and occupying empty slots on the game board above. The features of the token game, inherited from the properties of rolls, are as follows:

1. Each roll of the Roberts graph in a particular direction slides *all* the tokens along that direction one place up.

2. When a token reaches the end of its slide (eg, direction x_4, as displayed by the fourth column of Figure 10), it can no longer use that direction.

3. The antipode to the base (topmost on the Roberts graph) acts as a *transfer point*, moving tokens from one directional slide into another.

Theorem 5 *For any n-cube partition, there exists a path unfolding of an n-cube whose bounding box dimensions matches the partition.*

Proof. We provide an unfolding algorithm by rolling along directions satisfying a given partition. Parts in the partition with more than one token are called *towers*, whereas parts with exactly one token are dubbed *singletons*. Begin by decomposing the $2n-1$ tokens into four groups:

1. The set S of tokens in the singletons.

2. The set B of bottom tokens in each tower.

3. The set T of top tokens in each tower.

4. The remaining set M of (middle) tokens.

It follows that $|T| = |B| = (n-1) - |S|$ and

$$|M| = (2n-1) - |T| - |B| - |S| = |S| + 1.$$

Example 5 *Figure 11 shows two distinct partitions of 15 tokens into 7 parts (when $n = 8$), labeled according to the terminology above. In these cases, it is clear that $|T| = |B|$ and $|M| = |S| + 1$.*
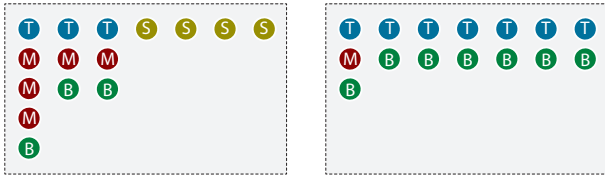

Figure 11: Two distinct partitions when $n = 8$.

Our algorithm is broken into three steps:

Step 1: *Perform one slide in each direction of a token from B.* This is possible since the transfer point is empty; see Figure 12(abc).

Step 2: *Perform alternating slides between tokens from M and S, starting and ending with M, until all such tokens depleted.* This is well-defined since $|M| = |S| + 1$. Since the first position on the game board along any element of M already contains a token from Step 1, a slide along its direction moves this token into the transfer point; see Figure 12(d). Now, sliding a token of S fills the first and last positions along this directional track with tokens, making this direction unusable; see Figure 12(e). This is ideal, for S contains only one token in each direction. After alternating between M and S, depleting all elements of S, slide one final time along the last element of M, loading a token onto the transfer point; see Figure 12(f).

Step 3: *Perform one slide in each direction of a token from T.* Each slide moves the token of the transfer point to the end of the track, which replenishing the transfer point with another token. This fills all the positions, as these are the final elements in each tower; see Figure 12(ghi). □

Observation 1 *Theorem 5 shows that the n-cube can be unfolded into extremes: a long thin $2 \times \cdots \times 2 \times (n + 2)$ box and a cubelike $3 \times \cdots \times 3 \times 4$ box, with a spectrum of sizes in between. It would be interesting to explore the* distribution *of cube partitions over all possible unfoldings of the n-cube.*

Observation 2 *Up to symmetry, there are 11 nets of the 3-cube and 261 nets of the 4-cube. For a general n-cube, it is an open problem to enumerate its distinct nets. The theorem above provides a (very weak) lower bound to this problem.*
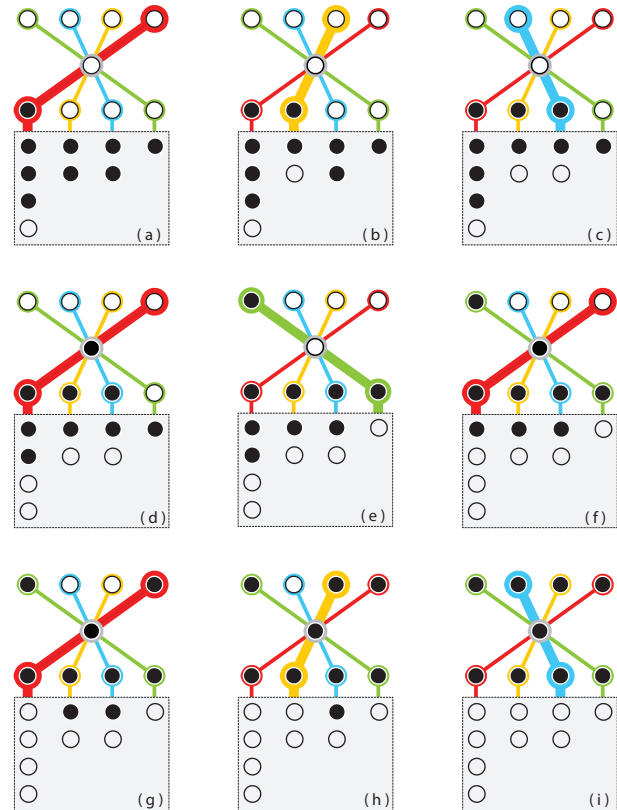

Figure 12: The partition slide algorithm.

4 Conclusion

The work of Horiyama and Shoji [3] show that every edge unfolding of the five Platonic solids results in a net. The higher-dimensional analogs of the Platonic solids are the regular convex polytopes: three classes of such polytopes exist for all dimensions (simplex, cube, orthoplex) and three additional ones only appear in 4D (24-cell, 120-cell, 600-cell). We have considered all unfoldings of cubes, and a similar result for simplices easily follows. We are encouraged to claim the following:

Conjecture 1 *Every ridge unfolding of a regular convex polytope yields a net.*

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