

Cartography of Tree Space

SATYAN L. DEVADOSS AND OWEN SCHUH

ABSTRACT

How can vibrant, contemporary art be produced that deals with vibrant, contemporary mathematics? To address this question, a collaboration began between an artist (Schuh) and a mathematician (Devadoss), revolving around recent problems in phylogenetics and the space of evolutionary trees. The result was twofold: First, a triptych of paintings was created, using acrylic, graphite, watercolor and metal leaf, that focused on different navigations within this tree space. Second, a novel set of open mathematics problems was discovered solely as a result of this investigation.

In our Enlightenment world, the work of the mathematician and the visual artist are usually not only viewed as incompatible but also held in tension. Mathematics is attributed to Platonic conceptions, addressed only through the mind, focusing on abstract ideas and theoretical structures. The visual artist, on the other hand, is assumed to be relegated to works of the hands, dealing with the concrete and the tangible. Bertrand Russell, a preeminent philosopher and mathematician of the 20th century, exemplified this attitude when he wrote:

Remote from human passions, remote even from the pitiful facts of nature, the generations have gradually created an ordered cosmos, where pure thought can dwell as in its natural home, and where one, at least, of our nobler impulses can escape from the dreary exile of the actual world [1].

Here, the “ordered cosmos” is mathematics, which can escape the “dreary exile” of physical reality, the reality artists inhabit. Today, a century after these words were penned, Russell’s viewpoint not only lingers but continues to define much of our worldview: Ideas based on equations and symbols hold a greater sense of power than notions from the visual realm. The rise of careers and funding opportunities in the STEM fields only highlights this.

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There has been considerable effort over the past few decades for serious engagement between the two fields [2]. Credit should be given to the journal *Experimental Mathematics*, which started to bring these domains together in 1992. Most notably, the collaboration by Anderson et al. has helped to define the role of drawing as a common language between mathematics and art; these authors posit the source of mathematical creativity to be the (usually inner) dialogue between the “Thinker” and the “Drawer”: “The Thinker exists in the world of linear logical thinking. The Drawer operates in the world of the imagination and of inverse vision” [3].

It is in fact the tension between the two modes of knowledge that pushes them into new territory. Much fertile ground is still unexplored, however.

PLAYERS

A collaboration began between an artist (Schuh) and a mathematician (Devadoss), engaging in current research mathematics through the eyes of a contemporary visual artist. To succeed, this partnership needed to be *complete*, with the mathematician involved in the drawings and the artist involved in the mathematics. Satyan Devadoss (formerly of Williams College) is the Fletcher Jones Professor of Mathematics at the University of San Diego, where much of his work revolves around discrete arrangements and their underlying geometry. Owen Schuh is a visual artist currently based in Philadelphia (formerly in San Francisco) who strives to demonstrate the tension between physical medium and logical and algorithmic structures.

This joint venture took place during an 18-month timeframe, from September 2013 to February 2015, overlapping mostly with the time when Devadoss was on sabbatical leave as visiting faculty at Stanford University. Roughly the first six months were spent in understanding the goals of the project and choosing a point of collaboration. Devadoss assisted Schuh in identifying and understanding the relevant material. The second six months were spent at coffee shops and studios in the Bay Area, where both mathematics and visual

sketches would be dissected and scrutinized. A typical meeting involved the artist presenting drawings and visual analyses, as well as asking any questions that might come up in the process. A dialogue then ensued with on-the-spot sketches and questions flowing in both directions, with subsequent “homework” for both parties.

The final months focused on crafting the final details and formulating a unifying vision to the project, resulting in a triptych of paintings. The emphasis of each of these final renderings was arrived at through dialogue, although the particular visual attributes and layout were the work of the artist, with Devadoss assisting in some of the final drawing of the triptych. The resulting undertaken project revolved around certain unsolved problems dealing with shapes and structures motivated by phylogenetics. This work culminated in an exhibit [4] at Satellite Berlin—*Art in Collaboration*, 14 March–25 April 2015.

MATHEMATICS

A classical problem in computational biology is to reconstruct an evolutionary “tree of life” from the genetic information of a set of species. The tree here is an abstract one (sometimes called a *phylogenetic tree*), with a *root* at the bottom, branching out into *leaves*. Each leaf is identified with a

different species in our set. As an example, given a set of four species $\{1,2,3,4\}$, Fig. 1 shows the 25 distinct ways these species can be related in a phylogenetic tree structure. The top row of trees has a single internal edge, with the other rows having two internal edges.

But instead of looking at this *set* of trees, we wish to form a *space* of trees. Just as the set of numbers can be given a shape, namely in the form of the number line, the set of trees can be arranged to form a space: Trees are related by *collapsing* and *growing* the internal edges. Indeed, each internal edge of a tree is given a length, viewed as the measurement of evolutionary time.

Figure 2 shows a part of this construction: Notice that by collapsing the top internal edge of tree A, we obtain tree Y. One can then grow an internal edge in two possible ways, resulting in trees B and C on the right. Similarly, collapsing the bottom internal edge of tree A results in tree X, which is related to trees D and E. This figure only shows how seven trees (A, B, C, D, E, X, Y) are related; the full tree space, which relates all 25 distinct trees, is shown in Fig. 3, resulting in the famous Peterson graph [5]. The 15 edges match up with rooted trees with four leaves and two internal edges (such as A, B, C, D, E above), and its 10 vertices match up with rooted trees with four leaves and one internal edge (such as X and

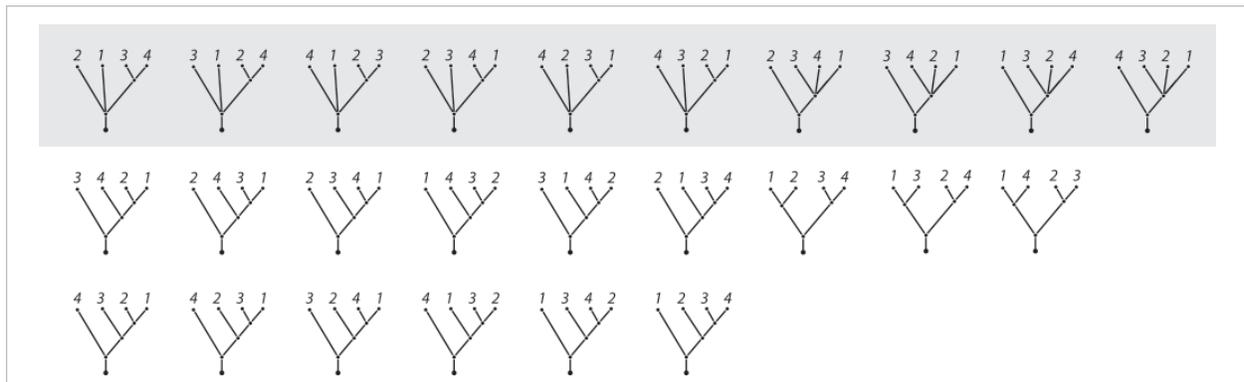


Fig. 1. Twenty-five tree-types with four species. (© Satyan L. Devadoss)

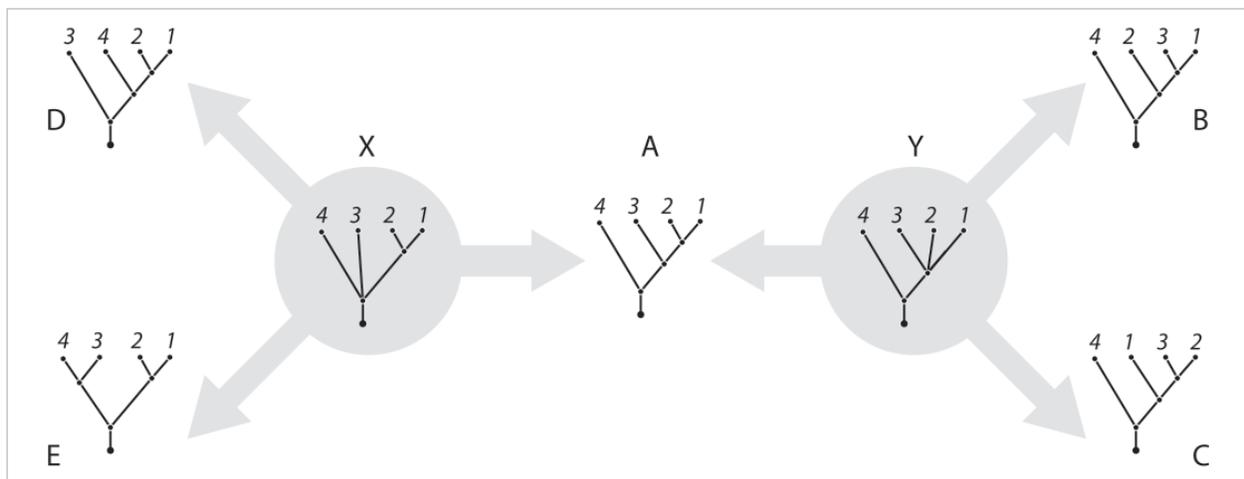


Fig. 2. Growing and shrinking edges in tree space. (© Satyan L. Devadoss)

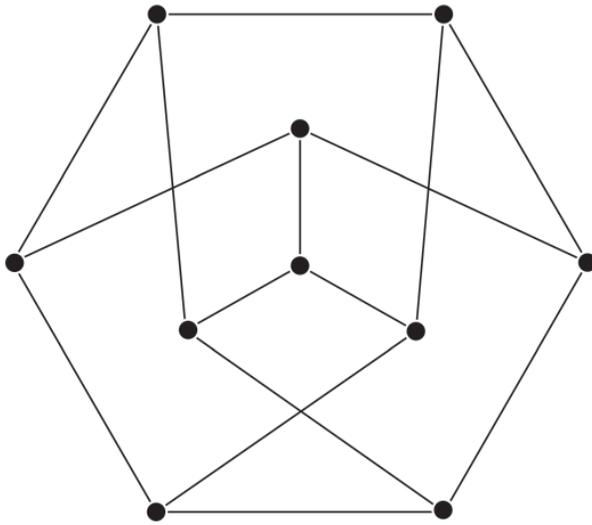


Fig. 3. Vertices and edges of tree space $T(4)$. (© Satyan L. Devadoss)

Y above). We denote $T(4)$ as the entire space of rooted trees with 4 leaves.

These tree spaces $T(n)$ exist for any value of n species, and have appeared in mathematics since the 1960s, under the theory of operads [6]. But recently, due to a surging growth in computational biology, tree spaces have resurfaced. This was spearheaded by a seminal work at the start of the 21st century by Billera, Holmes and Vogtmann [7], who studied the geometry of these spaces. However, understanding and visualizing these spaces is not only far from easy, but in many instances, quite novel.

FIVE SPECIES

Unlike the case above, the complications increase several-fold when considering $T(5)$, the space of tree for five species. This space is made of 105 triangles, 105 edges and 25 vertices, where each triangle corresponds to a tree with three internal edges. The combinatorial structure of these trees is given in Fig. 4: The top row shows trees with one internal edge, which come in three different types (K–M), with 10, 10 and 5 distinct trees of each type. These form the vertices of $T(5)$. Rows 2 and 3 show trees with two and three internal edges, respectively, coming in five (A–E) and three (X–Z) different types, forming the edges and triangles of $T(5)$. The connections between each of the rows shows adjacencies of the objects. The curious reader is encouraged to see Devadoss, Huang and Spadacene [8] for further details.

Visualizing only the vertices and edges results in Fig. 5a (drawn using Illustrator), where the five types of edges and the three types of vertices are color coded to match the scheme from Fig. 4. The rich, highly symmetric structure here is akin to the structures presented by Coxeter and Shephard [9]. Indeed, this deeper connection is not coincidental, as tree spaces $T(n)$ have close ties with objects called *Coxeter complexes* [10]. Note that including the 105 triangles of $T(5)$ in this diagram is not possible in two or three dimensions—that is, although this space is made of up flat triangular pieces, it needs higher dimensions to be accurately visualized. As an aside, $T(6)$ is formed by gluing together 945 distinct tetrahedra.

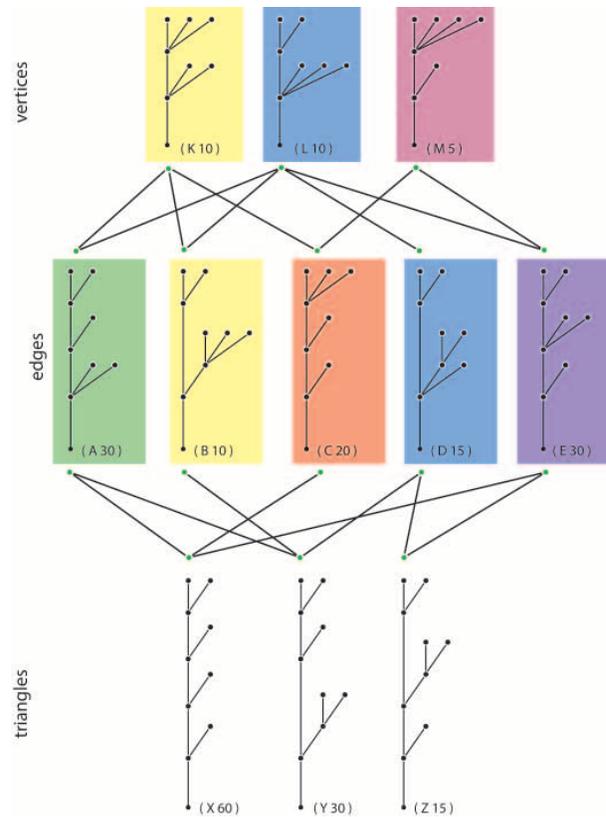


Fig. 4. Combinatorics and relationships between tree-types in $T(5)$. (© Satyan L. Devadoss)

TRIPTYCH

Our goal was not to describe the space in mathematical terms. Instead, we wanted to have the viewer experience *navigation* in this novel world: How would the inhabitants of $T(5)$ get around their land? How would different transportation systems exist and naturally interact with one another? In what ways would the different tree structures play a role in such systems?

The result was a triptych of paintings titled *The Cartography of Tree Space*: three panels, each created on a 108×108 cm panel. Each panel focused on a different “layer” of the transportation world and a different category of tree-type.

- The first painting (*Underground*, graphite, gouache, ink and acrylic: Fig. 5, top-right corner) targets the 25 vertices and reimagines them as stations in an underground system. Each vertex is a circle (corresponding closely to the vertices in Fig. 5a), and the edges of $T(5)$ appear as tunnels between them. These edges are limited to horizontal, vertical and 45-degree slants, as is common with subway maps. The tips of these tunnels, as they appear on the circle, are consistently color coded with the schemata in Fig. 4.
- The second piece (*Woven*, gouache, acrylic, gold, copper and silver leaf: Fig. 5, bottom-left corner) focuses on the 105 edges as roads existing on the surface of

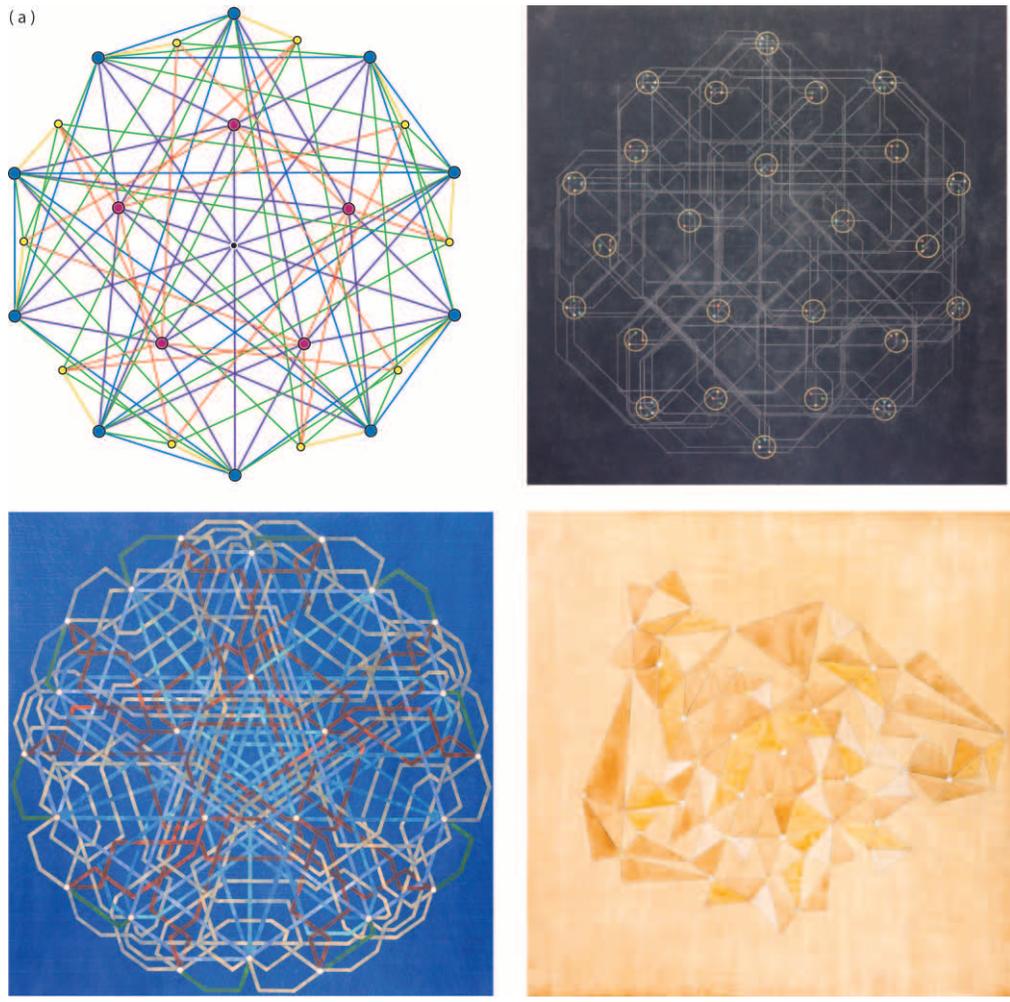


Fig. 5. Top row: Illustrator rendering of vertices and edges of $T(5)$; *Underground* wood panel. Bottom row: *Woven* wood panel; *Unfolded* wood panel. (Top left: © Satyan L. Devadoss. Top right, bottom row: © Owen Schuh.)

this world. An emphasis is placed on maintaining a five-fold symmetry as much as possible. The vertices are represented as pentagons and are leafed with gold, copper and silver, corresponding to distinct tree-types. Moreover, coincidences and superpositions of individual line segments are avoided.

- The third painting (*Unfolded*, acrylic, ink, tea, gold, copper and silver leaf: Fig. 5, bottom-right corner) targets the 105 triangles presented from a vantage point in the skies. Here the patches of land are shaded one of three colors (60 brown, 30 golden and 15 white triangles), corresponding to the three types of tree-types in Fig. 4. The metal leafing of the vertices also corresponds to Fig. 5a.

In order to keep from “getting lost” in the details and complexities, the artist relied on a set of procedures to keep track of his own location in tree space. These procedures determined the order in which the image was drawn, the angles of the lines, how they cross and, less formally, “Lines connect between vertices in an elegant and direct manner.” In the first attempts, this meant drawing out all the permutations and then numbering them and listing each tree’s connections. The

drawing proceeded down the list until it was complete, and a rather tangled mess emerged. Later attempts made use of the patterns and symmetries, which became apparent in this first attempt. A different procedure would yield an equally valid rendering, with a possibly radical look.

While a computer model would have been possible for this project, we felt that drawings by hand were more appropriate. The computer changes the way one interacts with the object: Although easier to manipulate and alter, this behavior in and of itself could just as surely cause certain aspects to be overlooked. Shoehorning tree space into a two-dimensional frame that takes a longer time to “render” forces one to slow down and perhaps approach it differently.

CONCLUSION

The mathematical definition of a line has neither thickness nor color, but in drawing there is no line without these characteristics. As mathematics tries to create abstractions from reality, these works show alternate realisms arising from such abstractions. The careful selection of color and line places the artwork in relation to other images, forms and ideas, and creates openings for the viewer to visually experience the structure of tree space. Although these decisions may seem arbitrary relative to the mathematics, they are necessary for

the realization of any visual representation at all. While a completed drawing can be apprehended “all at once” as a whole, it still must be drawn one line at a time, similar to the distinction between mathematical insight and the rigor of a proof. The resulting art is not intended to decorate or obfuscate the mathematics; rather it makes visible the work of understanding and rendering a representation of the math object in physical form.

At the end of this process, the mathematician had more to say about the art, and the artist had more to inquire about the mathematics. Not only were three visual pieces produced, but several mathematics questions arose during the process. (The diligent reader is encouraged to pursue Devadoss et al. [11] for the necessary background.) In particular, the art led to these open questions:

- Expanding on the notion of maps and cartography, what can be said about distances between two points in tree space? What about certain restrictions that limit the inhabitant to avoid or use only certain tree-types of triangles or edges? How can efficiency of travel in tree space be measured with different weights placed on different tree-types?
- What is the least number of chambers (in the form of *associahedra*) needed to cover the tree space $T(n)$? We know $n!/2$ such chambers is enough, but can this

number decrease? In what ways would this help to form better maps of tree space?

- Another chamber-type (*permutohedra*) is obtained by permuting all the leaves of a tree in the shape of a caterpillar [12]. What chambers do we obtain when leaves of other tree-types are permuted? Can different navigation maps be created using different chambers of tree space? And what type of information would be needed to move from navigation maps that use different chamber types?

As the triptych paintings form one result of the collaboration, this set of open mathematics questions forms another. In all of this, the process behind the mathematics and the art was quite similar: Ideas were conjectured, tested and evaluated, both visually and analytically. And there was a sense of incredible freedom to explore these worlds, with a strong instinct guiding the collaborators as to the right road to pursue. Cambridge mathematician John Littlewood, in his wonderful 1953 manuscript, commented on the modern viewpoint of the duality between images and mathematical theory: “A heavy warning used to be given that pictures are not rigorous; this has never had its bluff called and has permanently frightened its victims into playing for safety” [13].

Our work is a testament towards bridging these two worlds together.

Acknowledgments

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SATYAN DEVADOSS received his PhD in mathematics from Johns Hopkins and became a full professor at Williams. He has held visiting positions at Ohio State; University of California, Berkeley; Harvey Mudd; Université Nice; and Stanford. Devadoss is a recipient of two national teaching awards as well as an inaugural Fellow of the American Mathematical Society, with support from the NSF, DARPA and the Mellon and Templeton Foundations.

Initially pursuing biology, **OWEN SCHUH** earned a degree in fine art and philosophy from Haverford College and received his Master’s of Fine Art from the Tyler School of Art. He returned to Haverford to teach drawing and painting before moving to San Francisco. He has exhibited in Germany, Italy and throughout the United States, and his work is included in a number of private collections.