Unfolding the Simplex and Orthoplex

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Abstract

Over a decade ago, it was shown that every edge unfolding of the Platonic solids was without self-overlap, yielding a valid net. We consider this property for regular polytopes in higher dimensions, notably the simplex, the cube, and the orthoplex. It was recently proven that all unfoldings of the n-cube yield nets. We show that this property holds for the n-simplex and the 4-orthoplex but fails for any orthoplex of higher dimension.

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1 Introduction

The study of unfolding polyhedra was popularized by Albrecht Dürer in the early 16th century who first recorded examples of polyhedral *nets*, connected edge unfoldings of polyhedra that lay flat on the plane without overlap. Motivated by this, Shephard [7] conjectures that every convex polyhedron can be cut along certain edges to admit a net. This claim remains tantalizingly open and has resulted in numerous areas of exploration.

We consider this question for higher-dimensional *polytopes*: The codimension-one faces of a polytope are *facets* and its codimension-two faces are *ridges*. The analog of an edge unfolding of polyhedron is the *ridge unfolding* of an n-dimensional polytope: the process of cutting the polytope along a collection of its ridges so that the resulting (connected) arrangement of its facets develops isometrically into an $\mathbb{R}^{n-1}$ hyperplane. In our work, instead of trying to find one valid net for each convex polyhedron (as posed by Shephard), we consider a more aggressive property:

▶ **Definition.** *A polytope P is* all-net *if every ridge unfolding of P yields a valid net.*[1]

A decade ago, Horiyama and Shoji [4] showed that the five Platonic solids are all-net. Figure 1 shows the 11 different unfoldings (up to symmetry) of the octahedron, all of which are nets. The higher-dimensional analogs of the Platonic solids are the regular polytopes. Three classes of regular polytopes exist for all dimensions: the n-simplex, n-cube, and n-orthoplex.[2] It was recently shown that the n-cube is all-net [2]. We prove that the n-simplex and 4-orthoplex are as well. Surprisingly, for all $n > 4$, the n-orthoplex fails to be all-net.

▶ Remark. Sam Zhang [9] has built a lovely interactive software that creates every net of the 4-cube, 4-simplex, and 4-orthoplex by drawing spanning trees on its dual 1-skeleton.

[1] This nomenclature comes from Joe O'Rourke: a basketball "all-net" shot scores by not touching the rim, as all unfoldings become successful nets by facets not overlapping and touching each other.
[2] The orthoplex is dual to the cube and is sometimes called the *cross-polytope* or the *cocube*.

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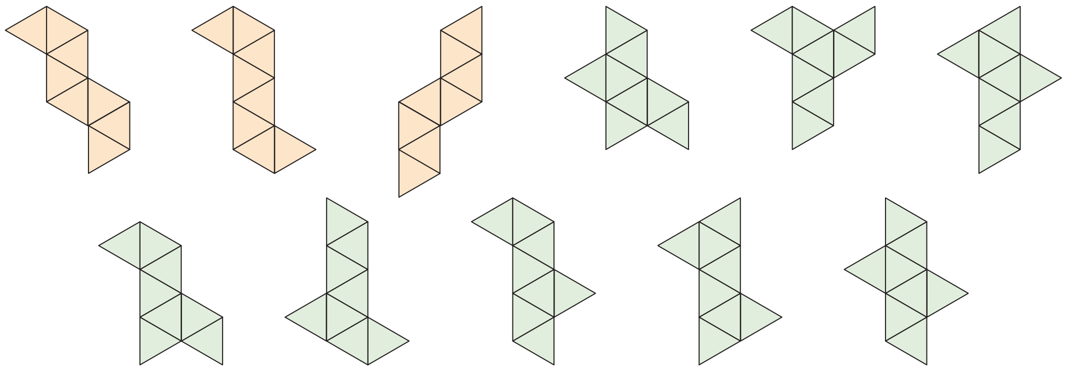

Figure 1 The 11 nets of the octahedron, also known as the 3-orthoplex.

2 Unfolding the Simplex

We explore ridge unfoldings of a convex polytope P by focusing on the combinatorics of the arrangement of its facets in the unfolding. In particular, a ridge unfolding induces a spanning tree in the 1-skeleton of the dual of P: a tree whose nodes are the facets of the polytope and whose edges are the uncut ridges between the facets [6]. Our focus throughout this paper will be on the n-simplex and the n-orthoplex, both of whose facets are $(n-1)$-simplices. First, we study paths in the 1-skeleton, corresponding to a chain of unfolded simplicial facets.

▶ **Definition.** *A* list $\mathcal{L} = \langle a_1, a_2, \ldots, a_k \rangle$ *is a sequence of numbers from* $\{1, \ldots, n\}$ *(possibly with repeats) where no number is listed twice in a row.*

Label the vertices of the $(n-1)$-simplex S with the numbers $1, \ldots, n$. Given a list $\mathcal{L}$ with k elements, we construct a chain $C(\mathcal{L})$ of $k+1$ simplices from the list as follows: Starting with $S = S_1$, attach a simplex S_2 to S_1 on the facet of S_1 that is opposite vertex a_1. Note that all but one of the vertices of S_2 will inherit a label from S_1 and we label the remaining one a_1. Attach a third simplex S_3 to S_2 on the facet opposite vertex a_2, and extend the labeling from S_2 to S_3 as before, and continue in this matter until the list is exhausted. Figure 2 shows this process in action for the list $\langle 3, 2, 3 \rangle$, creating a chain of four 2-simplices.

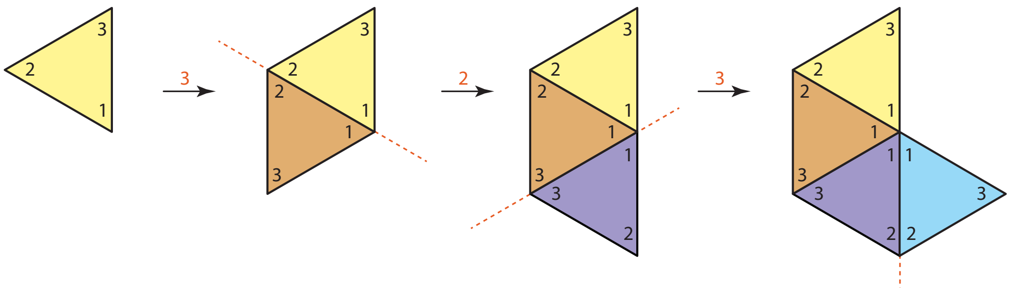

Figure 2 The chain of simplices assembled from the list $\langle 3, 2, 3 \rangle$.

We now introduce a coordinate system to capture the geometry. Begin by placing the n vertices of the $(n-1)$-simplex S at the standard basis vectors e_i of $\mathbb{R}^n$. Note that the coordinates of its vertices are recorded as the column vectors of the $n \times n$ identity matrix. The rest of the chain is then placed in the hyperplane $x_1 + \cdots + x_n = 1$ by a sequence of reflections. Let ρ denote the reflection of S across its facet opposite the vertex (say v) labeled with number a_1. Thus, ρ fixes all vertices except for v; see Figure 3.

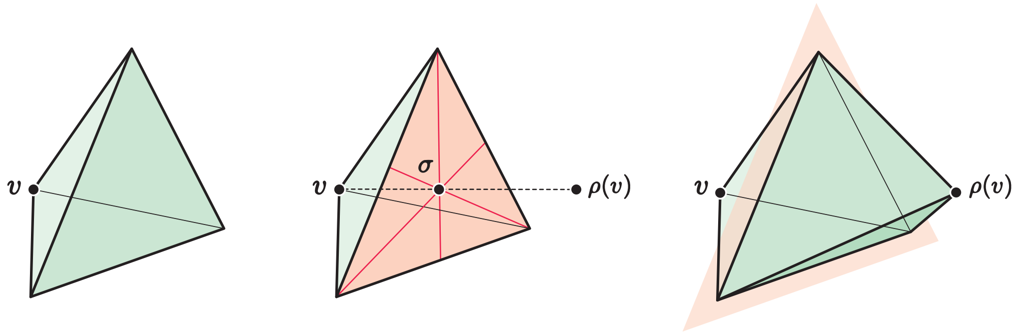

Figure 3 The reflection of the vertex across the opposite face.

To calculate the coordinate of $\rho(v)$ in $\mathbb{R}^n$, we first find the center σ of the facet opposite v, given by $\sigma = 1/(n-1) \cdot (1, \ldots, \ 0, \ \ldots, 1)$, where 0 occurs in the a_1-th coordinate. Since σ bisects the segment from v to $\rho(v)$,

$$\rho(v) = \ v + 2\ \overrightarrow{v\,\rho(v)} \ = \ (0, \ldots, 1, \ldots 0) \ + \ 2\left(\frac{1}{n-1}, \ldots, -1, \ldots, \frac{1}{n-1}\right),$$

where the -1 occurs in the a_1-th coordinate. Hence the reflection ρ is given by a matrix M_{a_1}, which is the identity except for $\rho(v)$ in the a_1-th column. Thus the coordinates of the i-th vertex of S_2 are recorded in the i-th column v_i of $N_1 = M_{a_1}$. By change of coordinates, its image under the reflection from S_2 to S_3 is

$$N_1 M_{a_2} N_1^{-1} v_i = N_1 M_{a_2} e_i\,,$$

and thus, the coordinates of the i-th vertex of S_3 are recorded in the i-th column of $N_2 = N_1 M_{a_2}$. Note that because M_{a_2} affects only the a_2 column, N_1 and N_2 differ only in the a_2 column. Continuing in this way, the vertices of S_{k+1} are recorded as the columns of $N_k = N_{k-1} M_{a_k}$.

An n-simplex has $n+1$ facets, and each is adjacent to every other. Thus, any listing of the facets (without repeat) describes a chain. However, because the full symmetric group acts transitively on the simplex, there is essentially only one chain, say $\langle 1, 2, \ldots, n \rangle$. Since the first facet is exactly the portion of $\sum x_i = 1$ that lies in the first orthant, a subsequent facet will only intersect the first if it contains a point that has all positive coordinates. This never happens, and a detailed proof is given in [3, Section 2.3]. Hence:

▶ **Theorem 1.** *Every unfolding of the n-simplex yields a net.*

3 Orthoplex Combinatorics and Geometry

In contrast to the simplex, both the unfoldings of the n-orthoplex and the chains within these unfoldings exhibit considerable variety. Unfoldings of the n-orthoplex are in bijection with spanning trees of the 1-skeleton of the n-cube. Consider the following approach to record paths on this skeletal structure: Position the n-cube with antipodal vertices at $(0, \ldots, 0)$ and $(1, \ldots, 1)$. A path along the edges of this cube is encoded as a list of binary numbers (sometimes called a *Gray code*) where exactly one digit changes from one entry to the next. For our work, our list $\mathcal{L}$ simply records the digit entry that changes in moving from one vertex to another. By duality, the ridges of the orthoplex inherit these labels and the process of unfolding the chain corresponds to the construction of $C(\mathcal{L})$.

▶ **Example.** *Consider the Gray code* $\langle 101, 100, 110, 111\rangle$ *associated to the list* $\langle 3, 2, 3\rangle$*. Figure 4(a) shows the path on four vertices of the cube, (b) corresponding to four adjacent facets of the octahedron, (c) resulting in a partial chain unfolding. Compare to Figure 2.*

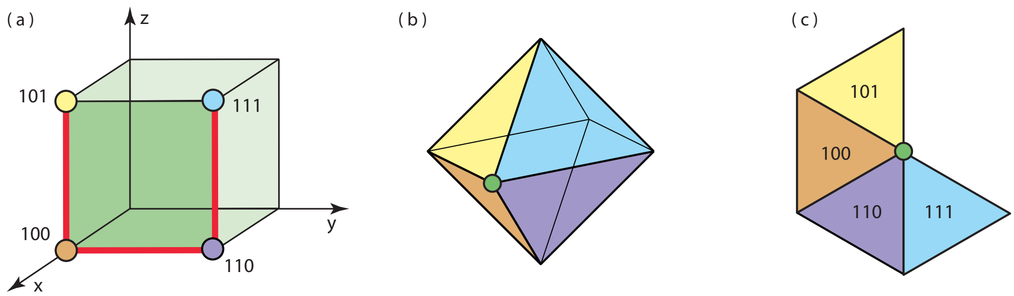

Figure 4 Path on the 3-cube and a partial unfolding of the octahedron.

▶ Remark. Up to symmetry, there are just three spanning paths on the 1-skeleton of the 3-cube: $\langle 1, 2, 1, 3, 1, 2, 1\rangle$, $\langle 1, 2, 1, 3, 2, 1, 2\rangle$, and $\langle 1, 2, 3, 2, 1, 2, 3\rangle$, corresponding to the first three highlighted nets shown in Figure 1. The situation escalates rapidly as n increases: there are 238 spanning paths on the 4-cube and 48,828,036 on the 5-cube [5].

▶ **Definition.** *A list of numbers from* $\{1, \ldots, n\}$ *is* valid *if it corresponds to a path on the* n*-cube.*

A list is valid as long as the route it describes on the cube does not cross itself, which can be characterized as follows:

▶ **Lemma 2.** *A list is valid if and only if it contains no sublist of consecutive entries in which each entry occurs an even number of times.*

▶ Remark. With this characterization, it is straightforward to create an algorithm to build valid lists: recursively append numbers $\{1, \ldots, n\}$ and check whether any of the new consecutive sublists have entries that occur an even number of times.

The question of whether two facets overlap depends on how close they are to each other, which can be estimated by calculating the distance between their centroids. If the vertices are $v_i = (a_{i1}, \ldots a_{in})$, the centroid is found by averaging their coordinates:

$$\left(\frac{1}{n} \sum a_{1j}, \; \ldots, \; \frac{1}{n} \sum a_{nj} \right).$$

It is straightforward to calculate the necessary distances:

▶ **Lemma 3.** *Let* d *denote the distance between the centroids of two* $(n-1)$*-simplex facets of the* n*-orthoplex in an unfolding. If* $d < 2/\sqrt{n(n-1)}$*, the facets must intersect. If* $d > 2\sqrt{(n-1)/n}$*, the facets cannot intersect.*

4 Orthoplex Unfolding

This section proves that the 4-orthoplex is all-net. We do this by extending paths on the skeleton of the 4-cube. While any path along a 3-cube can always be extended to a spanning path, this is not true for $n \geq 4$. For example, Figure 5(a) shows the 1-skeleton of the 4-cube, and the blue path shown in (b) cannot be extended further.

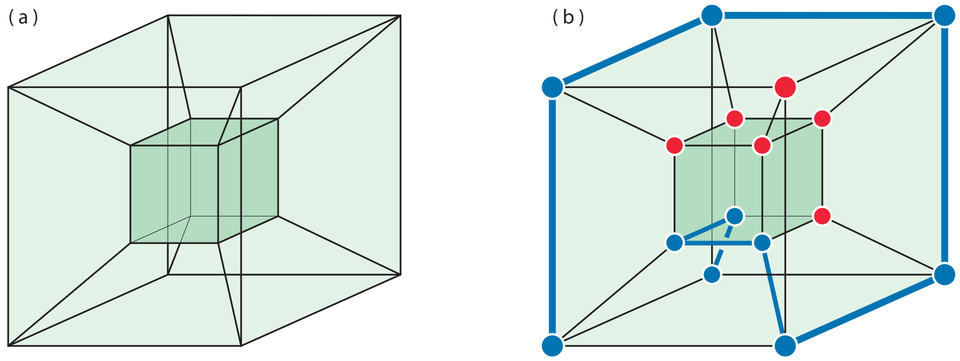

Figure 5 A path in the 4-cube that cannot be further extended.

Notice that a path can no longer be extended only when it has already crossed through all vertices adjacent to its two endpoints. Each vertex of the 4-cube is adjacent to four others, so roughly speaking, we might expect a path to pass through eight additional points before reaching its end. It is not quite this simple, because some of these points may overlap, but by considering the possible configurations, we arrive at a slightly weaker result.

▶ **Lemma 4.** *A path on the skeleton of the* 4*-cube can be extended to connect at least nine vertices.*

Rephrasing Lemma 4, any valid list can be extended to a valid list with at least eight entries. There are relatively few valid lists with eight entries, and by direct inspection it can be seen that they all yield nets, so:

▶ **Lemma 5.** *Every valid list containing exactly eight entries unfolds to form a partial net of the* 4*-orthoplex.*

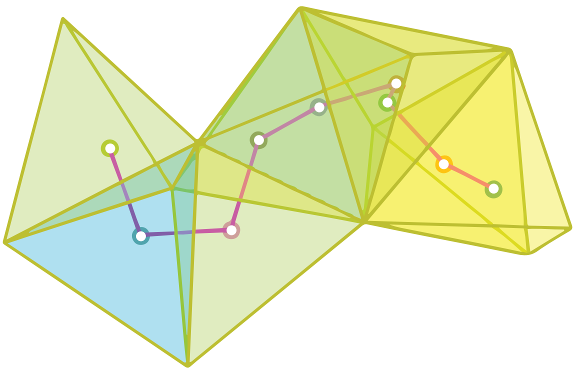

Figure 6 The partial unfolding corresponding to a valid list with length eight.

In unfoldings corresponding to longer lists, individual facets may be separated by more than eight facets. In these cases, we can calculate the distance between centroids. In every case, the distance is large enough to guarantee that the facets do not intersect, so:

▶ **Lemma 6.** *If two facets of the* 4*-orthoplex are separated by eight or more facets, they cannot overlap.*

Buekenhout and Parker [1] enumerate 110,912 ridge unfoldings of the 4-orthoplex. The following guarantees that they are all valid nets.

▶ **Theorem 7.** *The 4-orthoplex is all-net.*

Proof. If there were an unfolding that did not yield a net, then there would be a path between two of its overlapping facets. By Lemma 6, those facets must be separated by fewer than eight intervening facets along the path, corresponding to a valid list $\mathcal{L}$ whose length is at most eight. By Lemma 4, that list can be extended to one whose length is exactly eight. As described in Lemma 5, none of the unfolds generated by these lists exhibit overlap. ◀

Moving to higher dimensions, although the n-cube is all-net [2], its dual is not:

▶ **Theorem 8.** *For each $n > 4$, the n-orthoplex is not all-net.*

Proof. In dimensions 5 – 9, specific lists demonstrate overlap using centroid arguments:

$$\begin{array}{lcl} \text{dim. } 5 & : & \langle 1,2,3,4,2,1,5,4,2,4,5,4,2,1,5,4,3,1,5\rangle \\ \text{dim. } 6 & : & \langle 1,2,3,1,4,5,4,3,5,4,1,3,2,1,4\rangle \\ \text{dim. } 7,\, 8 & : & \langle 1,2,3,4,1,5,3,5,4,3,2,1\rangle \\ \text{dim. } 9 & : & \langle 1,2,3,4,2,4,1,2,3\rangle. \end{array}$$

It turns out that the dimension 9 example fails to unfold to a net for any $n > 9$. However, in higher dimensions, the centroid measurements become less robust. Instead, we return to the idea used in the simplex proof. It suffices to show that a point in the tenth facet has all positive coordinates. The point $v = 1/(n-1)\langle 1, 0, 1, 1, \ldots, 1\rangle$ is the midpoint of the ridge of the first facet. It can be shown that its image

$$M_1 \cdot M_2 \cdot M_3 \cdot M_4 \cdot M_2 \cdot M_4 \cdot M_1 \cdot M_2 \cdot M_3 \cdot v$$

in the tenth facet has all positive coordinates. Details are provided in [3, Section 4.3]. ◀

There are only three additional regular polytopes whose all-net property has not been studied, all of which are four-dimensional: the 24-cell, 120-cell, and 600-cell. The number of distinct unfoldings of these three polytopes are enumerated in [1]:

$$\begin{array}{rcl} \text{24-cell} & : & 6\,(2^{19} \cdot 5688888889 \;+\; 347) \\ \text{120-cell} & : & 2^7 \cdot 5^2 \cdot 7^3\,(2^{114} \cdot 3^{78} \cdot 5^{20} \cdot 7^{33} \;+\; 2^{47} \cdot 3^{18} \cdot 5^2 \cdot 7^{12} \cdot 53^5 \cdot 2311^3 \;+\; 239^2 \cdot 3931^2) \\ \text{600-cell} & : & 2^{188} \cdot 3^{102} \cdot 5^{20} \cdot 7^{36} \cdot 11^{48} \cdot 23^{48} \cdot 29^{30} \end{array}$$

The unfolding enumerations for these three exceptional polytopes encourage us to conjecture that all of them will fail to be all-net.

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