A SPACE OF CYCLOHEDRA

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ABSTRACT. The real points of the Deligne-Knudsen-Mumford moduli space $\overline{\mathcal{M}}_0^n$ of marked points on the sphere have a natural tiling by associahedra. We extend this idea to construct an aspherical space tiled by *cyclohedra*. We explore the structure of this space, coming from blow-ups of hyperplane arrangements, as well as discuss possibilities of its role in knot theory and mathematical physics.

To the memory of Rodica Simion.

1. ASSOCIAHEDRA AND CYCLOHEDRA

1.1. We present an overview of two important polytopes.

Definition 1.1.1. Let $\mathfrak{A}(n)$ be the poset of all dissections of a convex $(n + 1)$-gon using non-intersecting diagonals, ordered such that $a \prec a'$ if a is obtained from a' by adding new diagonals.[1] The *associahedron* K_n is a convex polytope of dim $n - 2$ whose face poset is isomorphic to $\mathfrak{A}(n)$.
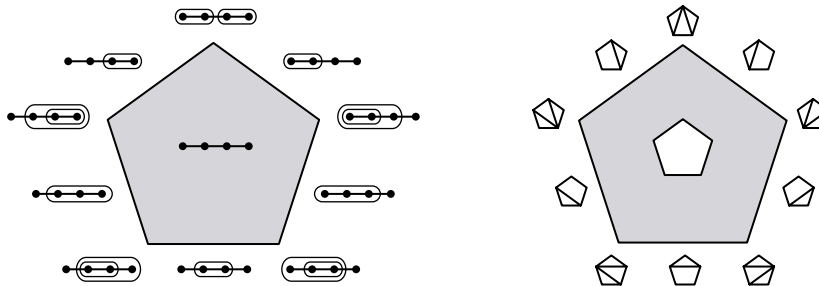

FIGURE 1. Associahedron K_4 using brackets and polygons.

The construction of the polytope K_n is given by Lee [14] and Haiman (unpublished). Stasheff originally defined the associahedron for use in homotopy theory in connection with associativity properties of H-spaces [18, §2]. There is a well-known bijection between dissections of a convex polygon and partial bracketings of letters. Figure 1 shows an example of this relationship.

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[1]Mention of *diagonals* will henceforth mean non-intersecting ones.

1.2. The cyclohedron W_n originally manifested itself in the work of Bott and Taubes and later given its name by Stasheff. A construction of the polytope W_n is given by Markl [15, §1] based on the following.

Definition 1.2.1. Let $\mathfrak{B}(n)$ be the poset of all partial bracketings of n letters arranged on a circle, ordered such that $b \prec b'$ if b is obtained from b' by adding new pairs of brackets. The *cyclohedron* W_n is a convex polytope of dim $n-1$ whose face poset is isomorphic to $\mathfrak{B}(n)$.

It was the clever idea of Simion to come up with an alternate poset isomorphic to $\mathfrak{B}(n)$; in fact, she provides a construction of W_n using this poset [17, §2]. It is formulated in terms of *centrally symmetric* $2n$-gons, where a (non-intersecting) diagonal on such a polygon will either mean a *pair* of *centrally symmetric* diagonals or a diameter of the polygon.

Proposition 1.2.2. *The poset of non-intersecting diagonals on a centrally symmetric $2n$-gon, ordered such that $b \prec b'$ if b is obtained from b' by adding (non-intersecting) diagonals, is isomorphic to $\mathfrak{B}(n)$.*

Figure 2 shows an example of the two descriptions of W_3. One difference between K_n and W_n is that for the codim one faces of the associahedron, we do not place brackets around all n variables. In contrast, the cyclohedron allows this since one can distinguish the cyclic manner in which the n variables are combined.
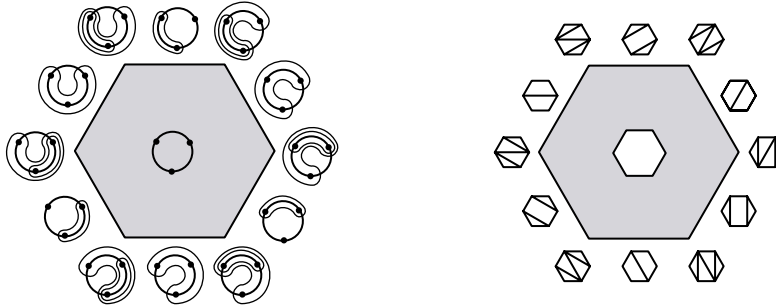

FIGURE 2. Cyclohedron W_3 using brackets and polygons.

1.3. It is worthwhile to explore some properties of W_n and its relationship with associahedra. It was noted by Stasheff that the boundary faces of K_n can be identified with products of lower dimensional associahedra [18, §2]. He makes a similar observation for W_n.

Proposition 1.3.1. [19, §4] *Faces of the cyclohedron are products of lower dimensional cyclohedra and associahedra. In particular, all codim k faces of W_n have the form*

$$W_{n_0} \times K_{n_1} \times \cdots \times K_{n_k},$$

over varying values of $1 \le n_0 < n$ and $1 < n_i \le n$ such that $\sum n_i = n + k$.

Example 1.3.2. Since W_5 is a four-dimensional polytope, we look at its possible codim one faces. They are given by the different ways of placing a diagonal in a 10-gon. Figure 3 illustrates the four possible types: On the far left is the product $W_4 \times K_2$; since K_2 is simply a point, the result is W_4 itself. The middle figures show $W_3 \times K_3$ and $W_2 \times K_4$, where on one hand the line segment is K_3 and on the other W_2. The last possible type is $W_1 \times K_5$, which is simply K_5.
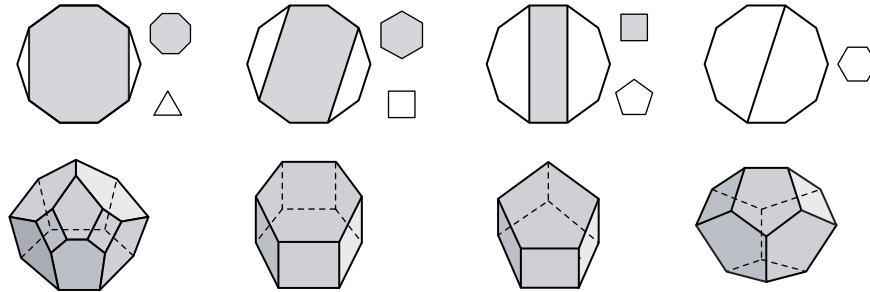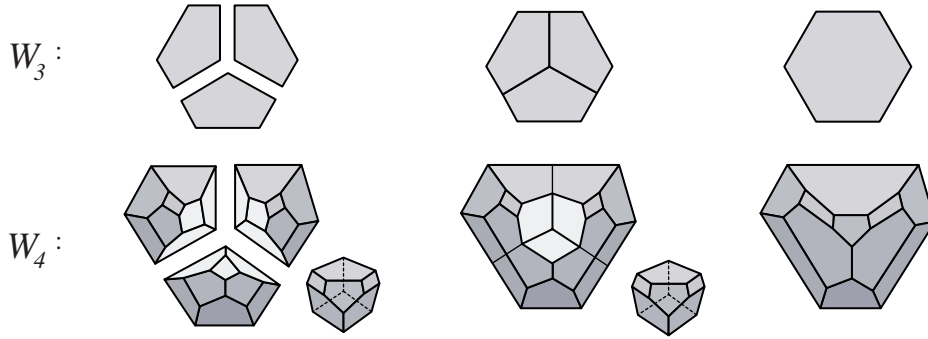

FIGURE 3. Codim one faces of W_5.

Remark. A few observations follow:

1. For the associahedron, it is well known that the inclusions of lower dimensional faces form the structure maps of an operad [18]. Stasheff and Markl show the inclusion maps above giving the cyclohedron a *right module* operad structure [15, §2].

2. The inclusion $K_n \hookrightarrow W_n$ coming from the proposition above shows the associahedron as a *face* of the cyclohedron. A. Tonks has constructed an explicit map $W_n \to K_{n+1}$ between polytopes of the *same* dimension [22], in the spirit of his map from the permutohedron P_n to K_{n+1} [21].

3. It is also noteworthy to point out a similar insight discovered by A. Ulyanov [23]. He shows n copies of K_{n+1} gluing together to form W_n. For example, the line segment W_2 is made up of two K_3 line segments with a pair of end points identified. Figure 4 shows the example of W_3 constructed using three copies of K_4, and similarly, W_4 from four copies of K_5. Note that this gluing involves some non-trivial smoothing of adjacent faces after the gluing.
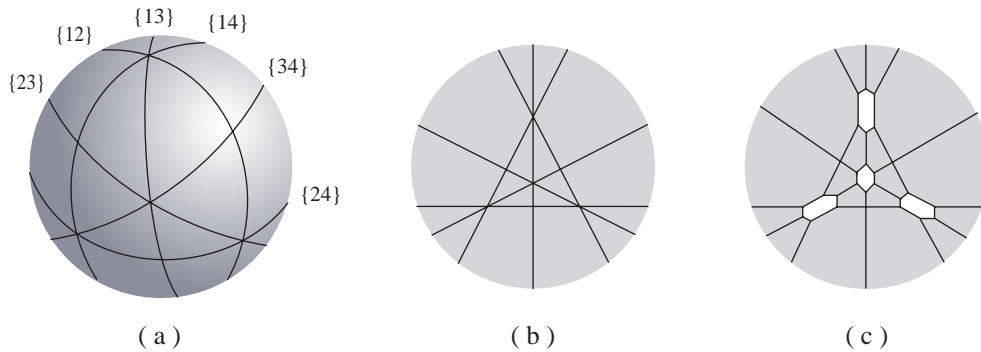
2. COXETER GROUPS AND BLOW-UPS

2.1. The relationship between Coxeter groups and the polytopes K_n and W_n is introduced here. We begin by looking at certain hyperplane arrangements and refer the reader to K.S. Brown [3] for any underlying terminology.

$W_3:$

$W_4:$

FIGURE 4. Building W_3 and W_4 from copies of K_4 and K_5.

Definition 2.1.1. For a Euclidean space V, a *linear hyperplane* is a codim one subspace $H \subset V$ passing through the origin. A *finite reflection group* W is a finite group of linear transformations of V generated by reflections over a set of linear hyperplanes.

Let $V^n \subset \mathbb{R}^{n+1}$ be the hyperplane defined by $\Sigma x_i = 0$. The collection of hyperplanes $\{x_i = x_j \mid i \neq j\}$ of V^n forms the *braid arrangement*. The symmetric group $\mathbb{S}_{n+1}$ is the reflection group acting on V^n, where the transposition (ij) acts as orthogonal reflections across the hyperplane $x_i = x_j$ by permuting the coordinates. Let $\mathbb{S}V^n$ and $\mathbb{P}V^n$ respectively be the sphere and the projective space in V^n; that is, $\mathbb{P}V^n$ is isomorphic to $\mathbb{RP}^{n-1}$. Note that the braid arrangement gives $\mathbb{P}V^n$ a CW-cellular decomposition into $\frac{1}{2}(n+1)!$ open simplices. Figure 5a and 5b depicts the $n = 3$ case.

{13}   {14}
{12}
{23}   {34}

{24}

(a)   (b)   (c)

FIGURE 5. $\mathbb{S}V^3$, $\mathbb{P}V^3$, and $\overline{\mathcal{M}}_0^5(\mathbb{R})$.

The Coxeter (or Weyl) group of type A_n is the symmetric group $\mathbb{S}_{n+1}$; moreover, it is the full symmetry group of the n-simplex, coming from representing the simplex as a convex linear combination of vectors. The Coxeter diagram

$$\bullet - \bullet \cdots \bullet - \bullet$$

with n nodes gives a presentation by n transpositions $\{s_i\}$, with the relations $(s_i s_{i+1})^3 = 1$ for adjacent generators and $(s_i s_j)^2 = 1$ otherwise. As discussed above, the braid arrangement dissects $\mathbb{P}V^n$ with the $(n-1)$-simplex as its fundamental chamber. The codim one faces of the simplex are indexed by the simple reflections, and hence by the nodes of the Coxeter diagram of A_n. The other faces are indexed by subdiagrams of the Coxeter diagram: a subdiagram gives a set of codim one faces, which intersect to give the indexed face.

Proposition 2.1.2. *Truncating faces of the $(n-1)$-simplex which correspond to connected subdiagrams of the Coxeter diagram of A_n in* increasing *order of dimension results in K_{n+1}. Furthermore, the associahedron is a simple polytope.*

The proof follows from the associahedron construction given in [7, §5]. Lee also obtains this using a sequence of stellar subdivisions on the polar dual to the associahedron [14, §3]. Note that the simplex is a simple polytope, and a truncation of a simple polytope remains simple. Figure 6 shows an example of the 3-simplex, with the initial truncation of vertices and then of edges, resulting in K_5.
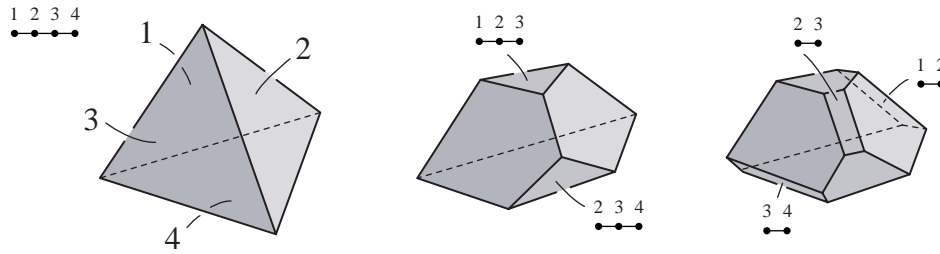

FIGURE 6. Truncation of simplex to K_5.

2.2. We move from an associahedron to a space tiled by them. Let M be a manifold and $D \subset M$ a union of codim one submanifolds which dissects M into convex polytopes (chambers). The collection of hyperplanes $\{x_i = 0 \mid i = 1, \ldots, n\}$ of $\mathbb{R}^n$ generates the *coordinate* arrangement. A crossing (of D) in M is *normal* if it is locally isomorphic to a coordinate arrangement. If every crossing is normal, then M is *right-angled*.[2] A construction which transforms any crossing into a normal crossing involves the algebro-geometric concept of a blow-up.

Definition 2.2.1. For a linear subspace X of a vector space Y, we *blow-up* $\mathbb{P}Y$ *along* $\mathbb{P}X$ by removing $\mathbb{P}X$, replacing it with the sphere bundle associated to the normal bundle of $\mathbb{P}X \subset \mathbb{P}Y$, and then projectifying the bundle.

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[2]The dual cellulation of M is a *cubical* cell complex: its cells are combinatorially equivalent to cubes.

A general collection of blow-ups is usually non-commutative in nature; in other words, the order in which spaces are blown up is important. For a given arrangement, De Concini and Procesi [6] establish the existence (and uniqueness) of a *minimal building set*, a collection of subspaces for which blow-ups commute for a given dimension, and for which the resulting space is right-angled. We use the definition found in [5, §3].

Definition 2.2.2. A codim k *minimal* element $\mathfrak{m}_k$ of the minimal building set of $\mathbb{P}V^n$ is the subspace indexed by a connected subdiagram of A_n with k nodes.

The Deligne-Knudsen-Mumford moduli space $\overline{\mathcal{M}}_0^n$ of marked points on the sphere is one of the central objects in mathematical physics. It is due to M. Kapranov that the connection between the braid arrangement and the real points $\overline{\mathcal{M}}_0^n(\mathbb{R})$ of this moduli space is made.

Theorem 2.2.3. [11, §4] *The iterated blow-up of $\mathbb{P}V^{n-2}$ along the cells $\{\mathfrak{m}_k\}$ in* increasing *order of dimension yields $\overline{\mathcal{M}}_0^n(\mathbb{R})$, with a tiling by $\frac{1}{2}(n-1)!$ copies of K_{n-1}.*

Example 2.2.4. Figure 5b and 5c shows $\mathbb{P}V^3$ before and after blowing up. The minimal elements of codim two turn out to be four points of triple intersection; blowing up along these components yield hexagons with antipodal identifications. Figure 5b has $\mathbb{RP}^2$ tiled by twelve simplices; Figure 5c is the connected sum of five real projective planes, tiled by twelve K_4 pentagons.

2.3. We move from *spherical* geometry coming from linear hyperplanes to *Euclidean* geometry arising from affine hyperplanes. Let $\widetilde{A}_n$ denote the *affine* root system of type A_n. The Coxeter diagram of $\widetilde{A}_n$ is $n+1$ vertices arranged on a circle and its Coxeter group is the semi-direct product $\widetilde{\mathbb{S}}_{n+1} = \mathbb{Z}^n \rtimes \mathbb{S}_{n+1}$, where the $n+1$ generators (transpositions) satisfy $(s_i s_j)^3 = 1$ when adjacent on the diagram and commute otherwise. Similar to above, let the nodes of the Coxeter diagram of $\widetilde{A}_n$ correspond to the top dimensional faces of the n-simplex.

Proposition 2.3.1. *Truncating faces of the n-simplex which correspond to connected subdiagrams of $\widetilde{A}_n$ in* increasing *order of dimension results in W_{n+1}. Furthermore, the cyclohedron is a simple polytope.*

The proof follows from the cyclohedron construction given in [19, Appendix B]. Figure 7 shows an example from $\widetilde{A}_3$, with the truncation of vertices and edges, resulting in W_4.

Definition 2.3.2. An *affine hyperplane* of V is a subset of the form $x + H$, where $x \in V$ and H is a linear hyperplane of V. An *affine reflection group* W is a locally finite group of affine isometries of V generated by reflections over a set of affine hyperplanes.

We follow the construction given in K.S. Brown [3, §6]. Let $V^n \subset \mathbb{R}^n$ be the hyperplane defined by $\Sigma x_i = 0$. The *affine* braid arrangement is the set $\{x_i = x_j + k \mid i \neq j,\ k \in \mathbb{Z}\}$ of
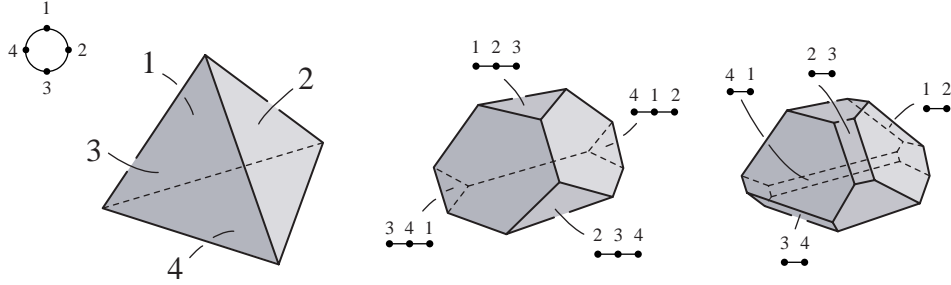
FIGURE 7. Truncation of simplex to W_4.

affine hyperplanes in V^n, having $\widetilde{\mathbb{S}}_n$ as its affine reflection group. This arrangement gives V^n a CW-cellular decomposition into an infinite collection of open $(n-1)$-simplices. The left side of Figure 8 gives the example when $n = 3$ generated by the reflections $\{(12), (23), (13)\}$.
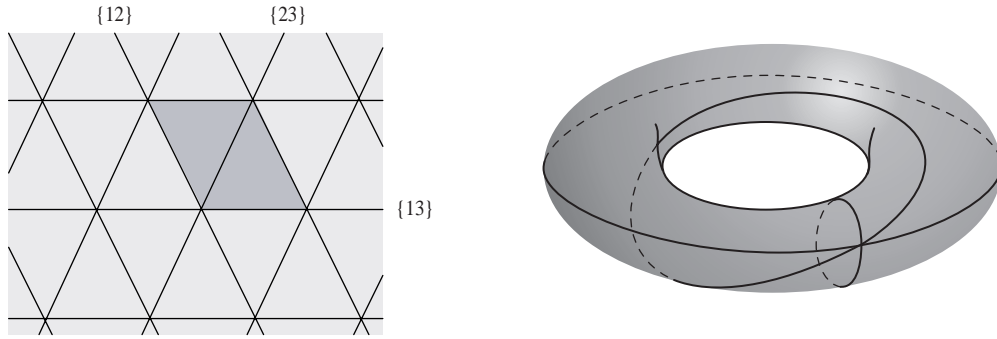

FIGURE 8. Tiling of $\mathbb{R}^2$ and $\mathbb{L}V^3$.

Previously, we looked to the projective sphere $\mathbb{P}V^n$ to give a compact space. In the affine case, the natural candidate is the quotient $\mathbb{L}V^n = V^n/\mathbb{Z}^{n-1}$, which is homeomorphic to the $(n-1)$-torus. Note that each chamber of V^n corresponds to an infinite, *cyclically repeating* chain of inequalities

$$\cdots < x_{i_n} + k_n - 1 < x_{i_1} + k_1 < x_{i_2} + k_2 < \cdots < x_{i_n} + k_n < x_{i_1} + k_1 + 1 < x_{i_2} + k_2 + 1 < \cdots,$$

where $k_i \in \mathbb{Z}$. Quotienting by $\mathbb{Z}^{n-1}$ amounts to only remembering the *cyclic* ordering $i_1 i_2 \cdots i_n$. Therefore, there are $(n-1)!$ chambers in $\mathbb{L}V^n$, each with a particular (cyclic) ordering. The right side of Figure 8 shows the resulting $\mathbb{L}V^3$ tiled by two simplices. The affine braid arrangement also has the notion of a minimal building set. Recall that each face of an $(n-1)$-simplex tiling $\mathbb{L}V^n$ is indexed by a subdiagram of $\widetilde{A}_{n-1}$.

Definition 2.3.3. A codim k *minimal* element $\mathfrak{m}_k$ of the minimal building set of $\mathbb{L}V^n$ is the subspace indexed by a connected subdiagram of $\widetilde{A}_{n-1}$ with k nodes.

Remark. There is an alternate, combinatorial formulation for $\mathfrak{m}_k$. A connected, length k subdiagram of $\widetilde{A}_{n-1}$ is the Coxeter diagram A_k having $\mathbb{S}_{k+1}$ as its Coxeter group. It follows that exactly $\binom{k+1}{2}$ hyperplanes intersect at $\mathfrak{m}_k$, with the transpositions of $\mathbb{S}_{k+1}$ generating reflections across these hyperplanes.

Definition 2.3.4. Let $\overline{\mathcal{Z}}^n$ be the iterated blow-up of $\mathbb{L}V^n$ along the cells $\{\mathfrak{m}_k\}$ in *increasing* order of dimension.

Corollary 2.3.5. $\overline{\mathcal{Z}}^n$ *is a manifold of dimension* $n-1$ *and is without boundary. Furthermore,* $\overline{\mathcal{Z}}^n$ *is tiled by* $(n-1)!$ *copies of* W_n *and is right-angled.*

Proof. These properties are immediate from the construction of $\overline{\mathcal{Z}}^n$ above and Proposition 2.3.1. An alternate proof of $\overline{\mathcal{Z}}^n$ being right-angled is given below. $\square$
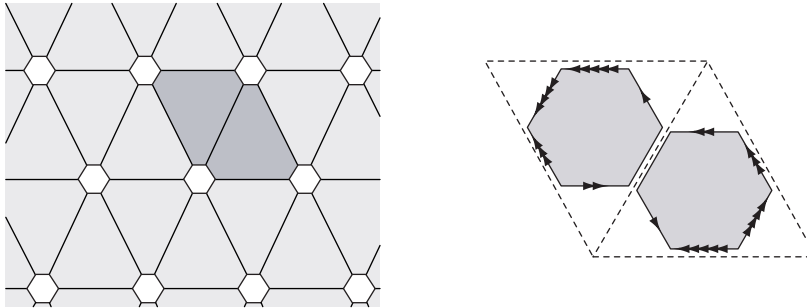

FIGURE 9. Hexagons tiling $\overline{\mathcal{Z}}^3$.

Example 2.3.6. The left side of Figure 9 shows the result after blowing up the minimal vertex of $\mathbb{L}V^3$. We see a tiling by two W_3 hexagons, illustrated in detail on the right of the figure. The resulting $\overline{\mathcal{Z}}^3$ manifold is homeomorphic to $\mathbb{RP}^2 \# \mathbb{RP}^2 \# \mathbb{RP}^2$.

3. Tilings and Compactifications

3.1. It was the combinatorial construction of $\overline{\mathcal{M}}_0^n(\mathbb{R})$ in [7, §3] which prompted Simion to wonder about a space of cyclohedra. Although we use her definition of the cyclohedron, our results are independent of her work.

Definition 3.1.1. The *twist* of a $2n$-gon G along a (pair of centrally symmetric) diagonal d, denoted by $\nabla_d(G)$, is the polygon obtained by separating G along d, twisting (reflecting) both pieces symmetrically, and gluing them back (Figure 10).

Label the edges of a $2n$-gon with $\{1, 2, \ldots, n\}$ such that antipodal edges have the same label. Identify two such labeled polygons by a rotation action, yielding $(n-1)!$ distinct polygons.
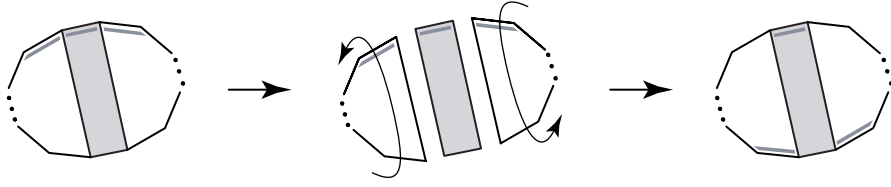
FIGURE 10. *Twist* along a diagonal.

Theorem 3.1.2. *Two centrally symmetric labeled $2n$-gons G_1, G_2 with k sets of diagonals (representing codim k faces of cyclohedra) are identified in $\overline{\mathcal{Z}}^n$ if there exist diagonals $d_1, \ldots, d_r$ of G_1 such that*

$$(\nabla_{d_1} \cdots \nabla_{d_r})(G_1) = G_2.$$

Proof. Recall that there are $(n-1)!$ chambers in $\mathbb{L}V^n$, each with a particular cyclic ordering $i_1 i_2 \cdots i_n$. We encapsulate this information by assigning to each chamber a centrally symmetric $2n$-gon, with sides labeled by the ordering $i_1 i_2 \cdots i_n$. The iterated blow-up of $\mathbb{L}V^n$ gives a tiling by cyclohedra, where the faces of W_n correspond to labeled $2n$-gons with diagonals. It remains to be checked that the gluing of these cyclohedra from minimal blow-ups is the same as using the twist operation. Without loss of generality, look at the fundamental chamber of $\mathbb{L}V^n$ defined by the ordering $12 \cdots n$. Each node of the Coxeter diagram of $\widetilde{A}_{n-1}$ indexes a codim one face of this simplex, which belongs to a hyperplane of the form $x_j = x_{j+1}$. Let $\mathfrak{m}_k$ be a minimal cell in the boundary of this chamber. Since the subdiagram associated to $\mathfrak{m}_k$ is *connected*, this cell occurs at such intersections $x_i = x_{i+1} = \cdots = x_{i+k}$ of hyperplanes.[3] Indeed, blowing up the cell seeks exactly to resolve the *order* in which collisions occur at such intersections. Crossing from the fundamental chamber through $\mathfrak{m}_k$ into its antipodal one in the arrangement corresponds to reflecting the element $i(i+1) \cdots (i+k)$ in the (cyclic) ordering $12 \cdots n$. Blowing up the minimal cell $\mathfrak{m}_k$ identifies cyclohedral faces across the antipodal chambers. Indeed, our notation of labeled polygons with twisting along diagonals mimics gluing antipodal faces after blow-ups. $\square$

Example 3.1.3. Figure 11 demonstrates the construction of $\overline{\mathcal{Z}}^2$ from one copy of W_2 to form S^1. The right side of Figure 9 shows the gluing of two W_3 hexagons resulting in $\overline{\mathcal{Z}}^3$.
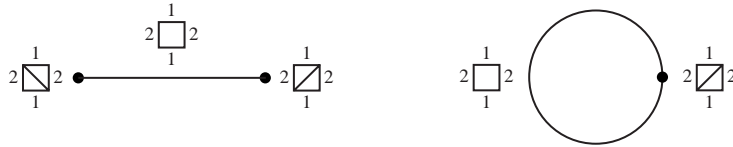

FIGURE 11. W_2 and $\overline{\mathcal{Z}}^2$.

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[3]Therefore, each $\mathfrak{m}_k$ corresponds to choosing $k+1$ elements from a set $\{x_1, \ldots, x_n\}$ of order n.

Corollary 3.1.4. $\overline{\mathcal{Z}}^n$ *is right-angled.*

Proof. Fix a codim k face of some cyclohedron in $\overline{\mathcal{Z}}^n$, represented by a labeled centrally symmetric $2n$-gon with k diagonals. Each twist along a diagonal moves to another cyclohedron adjacent to this face. There are 2^k such possible combinations of twists, giving a normal crossing at each face.   □

Remark. Faces of the associahedron are decomposed into the categories of dimension, type, and class, where they are used to better understand the structure of $\overline{\mathcal{M}}_0^n(\mathbb{R})$ (see [8, §2] for definitions and details). For example, the enumeration of faces with respect to dimension leads to the Euler characteristic of $\overline{\mathcal{M}}_0^n(\mathbb{R})$. Using methods and results from [8], the initiated reader could easily carry out similar combinatorial calculations for the cyclohedron to further describe properties of $\overline{\mathcal{Z}}^n$.

3.2. Our motivation in further analyzing $\overline{\mathcal{Z}}^n$ comes from some observations in Figure 9. The minimal vertex after blow-up becomes a hexagon with antipodal identification. This is essentially a triangle, which can be seen as $\overline{\mathcal{M}}_0^4(\mathbb{R})$. Note also that the three lines on $\mathbb{L}V^3$ become line segments with endpoint identifications in $\overline{\mathcal{Z}}^3$, which is seen from Figure 11 as three copies of $\overline{\mathcal{Z}}^2$. In general, we observe the following:

Theorem 3.2.1. *There are* $\binom{n}{k+1}$ *copies of* $\overline{\mathcal{M}}_0^{k+2}(\mathbb{R}) \times \overline{\mathcal{Z}}^{n-k}$ *in* $\overline{\mathcal{Z}}^n$.

Proof. We show that the blow-up of a minimal element $\mathfrak{m}_k$ of $\mathbb{L}V^n$ results in $\overline{\mathcal{M}}_0^{k+2}(\mathbb{R}) \times \overline{\mathcal{Z}}^{n-k}$. As discussed above, each $\mathfrak{m}_k$ corresponds to choosing $k+1$ elements from a set $\{1, \ldots, n\}$. Choose an arbitrary minimal cell and assign it such a choice, say $\{p_1, \ldots, p_{k+1}\}$. We view this as a centrally symmetric $2n$-gon having a diagonal partitioning it into a pair of symmetrically labeled sides $\{p_1, \ldots, p_{k+1}\}$, with the sides of the central polygon using the remaining labels. Note the correspondence of this dissected $2n$-gon to the product $K_{k+1} \times W_{n-k}$.

   As this minimal cell is blown-up, a system records the different ways points approach it; this corresponds to keeping track of the *order* of labels placed on the polygon. Since twisting is allowed along diagonals, there are $\frac{1}{2}(k+1)!$ ways to arrange labels on the non-central polygons, each corresponding to a K_{k+1}. Indeed, this is *exactly* how one gets $\overline{\mathcal{M}}_0^{k+2}(\mathbb{R})$, with the associahedra glued as defined above. Therefore, a fixed labeling of the central polygon gives $\overline{\mathcal{M}}_0^{k+2}(\mathbb{R}) \times W_{n-k}$, while allowing all possible labelings yields the result.   □

Example 3.2.2. We move to the $n = 4$ case to further illustrate the theorem. The left most drawing of Figure 12 shows the torus $\mathbb{L}V^4$ as the fundamental cube tiling $\mathbb{R}^3$ with hyperplane cuts. Blowing up the minimal vertex (codim three) and then the lines (codim two) yield the middle and the right hand pictures. The space $\overline{\mathcal{Z}}^4$ is shown to be tiled by six copies of W_4. Notice the vertices of the cube (identified to form a single vertex of the 3-torus) becomes $\overline{\mathcal{M}}_0^5(\mathbb{R})$ after blow-ups, tiled by twelve pentagons. The minimal lines

become triangular tori, that is, $\overline{\mathcal{M}}_0^4(\mathbb{R}) \times \overline{\mathcal{Z}}^2$. A face of the cube can be seen as a 2-torus with blow-ups, resulting in $\overline{\mathcal{Z}}^3$ tiled by two W_3 hexagons.
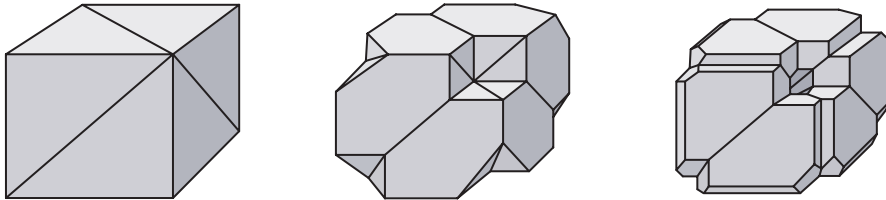

FIGURE 12. Blow-ups leading to six W_4 polytopes tiling $\overline{\mathcal{Z}}^4$.

3.3. The original definition of W_n describes it in terms of a compactified configuration space of points on the circle [2, §3]. We extend this to $\overline{\mathcal{Z}}^n$. A configuration space of n labeled points on an m-manifold X is defined as

$$\mathrm{F}(X,n) = X^n - \Delta, \quad \text{where } \Delta = \{(x_1,\ldots,x_n) \in X^n \mid \exists\, i,j,\ x_i = x_j\}.$$

Compactifying $\mathrm{F}(X,n)$ enables the points to collide and a *system* is introduced to record the *directions* points arrive at the collision. In the work of Fulton and MacPherson [9], this method is brought to rigor in the algebro-geometric context.[4] As points collide, they land on a *screen*, viewed as $\mathbb{RP}^m$, which is identified with the point of collision.[5] Now these points on the screen are themselves allowed to move and collide, landing on higher level screens. However, they are free up to an action of $\mathbb{PGl}_{m+1}(\mathbb{R})$, the affine automorphism on each screen. Kontsevich describes the process in terms of a magnifying glass: On any given level, only a configuration of points is noticeable; but one can zoom-in on a particular point and peer into its screen, seeing the space of collided points. As mentioned above, for an arrangement of hyperplanes, De Concini and Procesi develop a method to compactify their complements by iterated blow-ups using a minimal building set. In the case of the arrangement $X^n - \mathrm{F}(X,n)$, their procedure yields the Fulton-MacPherson compactification of $\mathrm{F}(X,n)$ [6].

Theorem 3.3.1. $\overline{\mathcal{Z}}^n$ *is a real Fulton-MacPherson compactification of* $\mathrm{F}(S^1,n)$ *quotiented by the action of* S^1.

Proof. The affine braid arrangement $\{x_i = x_j + k \mid i \neq j,\ k \in \mathbb{Z}\}$ decomposes $V^n \subset \mathbb{R}^n$ into $(n-1)$-simplices. Let H_i denote the hyperplane $x_1 = x_i$ and let V_i be the one-dim subspace of V^n defined by $H_2 \cap \cdots \cap \widehat{H}_i \cap \cdots \cap H_n$, where the symbol $\widehat{H}_i$ means the hyperplane H_i is to be deleted from the intersection. If v_i is a unit vector in V_i emanating from the origin,

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[4]Fulton-MacPherson is over the complexes, but for our purposes the reals are more relevant.

[5]These projective spheres have been dubbed *bubbles*, and the compactification process as *bubbling*.

then $\{v_i \,|\, i = 2, \ldots, n\}$ forms a set of $\mathbb{Z}$-basis vectors for a lattice. Without loss of generality, obtain the $(n-1)$-torus $\mathbb{L}V^n$ from V^n by modding out by this lattice.

On the other hand, since S^1 acts freely on $\mathrm{F}(S^1, n)$, its quotient simply fixes one of the n points, say $x_1 = 1 \in S^1$, resulting in $(S^1)^{n-1} - \Delta'$, where

$$\Delta' = \{(x_2, \ldots, x_n) \in (S^1)^{n-1} \mid \exists \, i, j, \ x_i = x_j \ \text{ or } \ x_i = 1\}.$$

It follows that the hyperplane markings on $\mathbb{L}V^n$ coincide with the arrangement Δ' on the $(n-1)$-torus. Therefore, from the discussion above, the Fulton-MacPherson compactification of $\mathrm{F}(S^1, n)/S^1$ coincides with the blow-ups of the minimal building set of $\mathbb{L}V^n$. $\square$

Remark. To see this for $n = 3$, note that $\mathrm{F}(S^1, 3)/S^1$, viewed as the front face of the cube in Figure 12, is identical to $\mathbb{L}V^3$, the shaded region of Figure 8.

The appearance of cyclohedra in the Fulton-MacPherson compactification above can intuitively be seen as follows: The compactification distinguishes the basespace S^1 from the bubbles $\mathbb{RP}^1$ coming from collisions. We keep track of this information by shading in the circle corresponding to the base space, shown in Figure 13a. There is a bijection between these bubble-trees and partial bracketings of n points on S^1, which can be expressed in terms of $2n$-gons with centrally symmetric diagonals (Figure 13b and 13c), demonstrating the appearance of W_n. Note how the collision of points is represented as drawing a diagonal *and* its symmetric counterpart. There is a $\mathbb{Z}_2$ reflection in $\mathbb{P}\mathrm{Gl}_2(\mathbb{R})$ acting on each $\mathbb{RP}^1$ bubble (coming from the affine automorphisms on each screen) which corresponds to twisting along diagonals of the $2n$-gons.
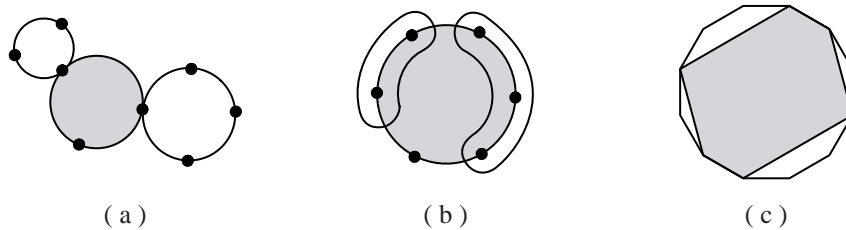

(a)   (b)   (c)

FIGURE 13. Bubbles and centrally symmetric polygons.

4. SOME OBSERVATIONS

4.1. We have shown the cyclohedron to be the $\widetilde{A}_n$ affine analogue of the associahedron. However, Simion thought of it as a type B_n associahedron. The reasoning behind this is explained by looking at the combinatorial question that motivated her. We thank V. Reiner for helpful insights into his work [16].

Definition 4.1.1. A k-*partition* separates $\{1, \ldots, n\}$ into disjoint sets $B_1, \ldots, B_k$. A partition is *non-crossing* if for elements $x_1, x_2 \in B_i$ and $y_1, y_2 \in B_j$ such that $x_1 < y_1 < x_2 < y_2$, then $i = j$. Let $NC(n, k)$ be the number of non-crossing k-partitions of $\{1, \ldots, n\}$.

The f-vector $\{f_i\}$ of a polytope P is a combinatorial invariant, where $f_k(P)$ equals the number of k-dimensional faces. Another method of encapsulating this information is by the h-vector $\{h_i\}$ (see [24] for details). Denoting K_n^* as the combinatorial dual of the associahedron, the following remarkable equivalence exists:

$$NC(n, n - k) = h_k(K_{n+1}^*).$$

The poset of partitions of $\{1, \ldots, n\}$ is isomorphic to the lattice of intersection subspaces for the type A_n braid arrangement. For example, a partition $\{1, 2, 4\}\{3, 5\}$ corresponds to the intersection subspace of $x_1 = x_2 = x_4$ and $x_3 = x_5$. Similarly, by replacing the set with $\{1, \ldots, n, \bar{1}, \ldots, \bar{n}\}$, we can consider the hyperoctahedral hyperplane arrangement for type B_n. The barring is an involution denoting the sign change coming from the hyperplanes $x_i = \pm x_j$.

Definition 4.1.2. A k_B-*partition* separates $\{1, \ldots, n, \bar{1}, \ldots, \bar{n}\}$ into disjoint sets B_0, B_1, $\ldots$, B_k, $\bar{B}_1$, $\ldots$, $\bar{B}_k$, where $\bar{B}_i = \{\bar{x} \mid x \in B_i\}$ and $B_i = \bar{B}_i$ only when $i = 0$.

Let $NC_B(n, k)$ be the number of non-crossing k_B-partitions of $\{1, \ldots, n, \bar{1}, \ldots, \bar{n}\}$. Simion was able to show that the cyclohedron satisfies the equality

$$NC_B(n, n - k) = h_k(W_{n+1}^*).$$

Remark. Combinatorially, Simion [17, §4] was able to construct a space tiled by eight copies of W_3 hexagons using the twist operation defined in [7]. We now see that she had arrived at the four-fold cover of $\overline{\mathcal{Z}}^3$ (in Figure 8, choose a region four times that of the shaded one). By labeling the antipodal sides of the polygons with i and $\bar{i}$, a 2^n-fold cover of the moduli space $\overline{\mathcal{Z}}^n$ is created.

Remark. Burgiel and Reiner discuss two more type B_n analogues of the associahedron, both being different from the cyclohedron [4].

4.2. The cyclohedron made its debut in knot theory implicitly through the work of Kontsevich [13] and explicitly by Bott-Taubes [2]. The imbedding $S^1 \to \mathbb{R}^3$ defining a knot induces a map onto their compactified n point configuration spaces, thereby introducing W_n. Roughly, one then 'carefully' integrates certain pullbacks of volume forms on S^2 using the inclusion $S^1 \to \mathbb{R}^3 \to S^2$; for a good topological understanding, see the work of D. Thurston [20].

The cyclohedron also appears in the work of Bar-Natan in a different setting [1, §4]. He gives a combinatorial approach of sketching some relations that arise in computing certain Kontsevich-KZ invariants coming from braids and chord diagrams (note that chord diagrams

come from $2n$ marked points on S^1 with pairwise identification). He introduces a multitude of relations and their corresponding polytopal structures, including W_n and Kapranov's permutoassociahedron KP_n.

4.3. If a manifold is right-angled, its dual cellation is a cubical cell complex. Gromov beautifully relates a combinatorial condition of such spaces with nonpositive curvature [10]. A simplical complex M is a *flag complex* if any finite collection of pairwise connected vertices spans a simplex in M.

Lemma 4.3.1 (Gromov). *The natural piecewise Euclidean metric on a cubical complex is nonpositively curved if and only if the link of every vertex is a flag complex.*

Gromov has also shown that the universal cover of a nonpositively curved space is contractible. In other words, the homotopy properties of these spaces are completely encapsulated in their fundamental groups. Davis, Januszkiewicz, and Scott apply Gromov's ideas to minimal blow-ups of certain hyperplane arrangements, proving the following:

Theorem 4.3.2. [5, §5] *The moduli spaces $\overline{\mathcal{M}}_0^n(\mathbb{R})$ and $\overline{\mathcal{Z}}^n$ are aspherical.*

Recall that the *Artin group* of A_n is the classic braid group on $n + 1$ strings between points on an interval (Figure 14a) or on a circle (Figure 14b); simply cut the circles open and lay the strings flat to obtain an isomorphism. The affine analog of classical braids is the Artin group of $\widetilde{A}_n$; this can be presented as braids with $n + 1$ strands restricted to a cylindrical shell. The latter diagrams in Figure 14 show two distinct elements of the $\widetilde{A}_2$ braid group. There is a close similarity between the fundamental group of $\overline{\mathcal{M}}_0^n(\mathbb{R})$ and the braid group, described from a combinatorial viewpoint in [7, §6]. It should be possible to extend this to the affine case.
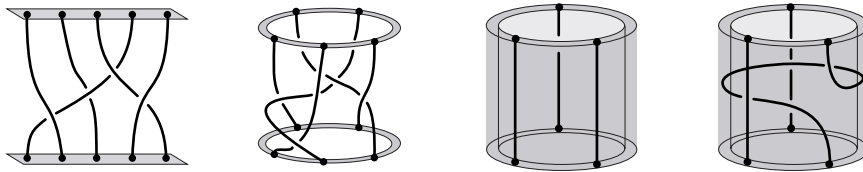

FIGURE 14. Four braids, two coming from A_4 and two from $\widetilde{A}_2$.

4.4. It seems quite natural to generalize what we have done. The motivation for the real moduli space of points $\overline{\mathcal{M}}_0^n(\mathbb{R})$ comes from its *complex* counterpart $\overline{\mathcal{M}}_0^n$. The importance of $\overline{\mathcal{M}}_0^n$ can be seen, for example, as being a fundamental building block in the theory of Gromov-Witten invariants, among numerous other appearances in literature. A reasonable candidate for the complex analogue for $\overline{\mathcal{Z}}^n$ could be the Fulton-MacPherson compactification of $F(S^2, n)/SO(3)$. Again, we would like to know the role this space would play in

mathematical physics and algebraic geometry. Similarly, we can generalize this construction to other Coxeter groups. Much has been done in terms of the minimal blow-ups in [5]. However, there have been a wealth of new compactifications that have been studied recently and we discuss the interplay between these ideas in a forthcoming work.

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